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Glueball spectrum from an effective hamiltonian

P.O. Hess[1,a], S.A. Lerma H.[2,b], J.C. López V.[2,b], C.R. Stephens[2,b], A. Weber[2,b]

[1] Institut für Theoretische Physik, Justus-Liebig-Universität, Heinrich-Buff-Ring 16, D-35392 Giessen, Germany
[2] Instituto de Ciencias Nucleares, UNAM, Circuito Exterior C.U., A.P. 70-543, Delegación Coyoacán, 04510 México D.F., Mexico

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Abstract. Using simple and general arguments we propose an effective Hamiltonian for the description of low-energy pure QCD. The Hamiltonian is a function of spatially constant collective modes. Its eigenstates can be organized into bands classified by the irreducible representations of an $O(8)$ group. The latter also determine parity and charge conjugation of the states. The energy spectrum agrees well with the glueball spectrum as measured on the lattice, and in particular the level ordering with respect to spin is naturally explained.

1 Introduction

QCD is widely considered to be the fundamental theory governing the strong interactions. Its chief success to date has been in describing experimental data at high energies – deep inelastic scattering – where the partonic nature of hadronic constituents plays a dominant rôle. In this regime the asymptotically free nature of the QCD running coupling constant, g, comes to the fore allowing for a perturbative expansion using as basis states free, i.e. $g \to 0$, gluons and quarks. At low energies, however, comparable succcess has been conspicuous by its absence. As is well known this is due to the phenomenon of infrared slavery which has as a consequence the confinement of coloured states, i.e. gluons and quarks are bound together to form hadrons. How to describe the crossover as a function of scale between such radically different effective degrees of freedom remains as one of the most challenging problems in particle physics.

Without reliable access to the infrared via the fundamental theory itself recourse is often made to treatments that use additional assumptions, such as adding confinement by hand (as in [1]–[6]), or that use effective field theory descriptions such as working directly with the meson and baryon fields and adjusting the interaction parameters to experimental data [7]. The fundamental question remains however: how does one derive the effective theory consistently from the underlying more fundamental theory? In the present case, as QCD has only one parameter, Λ_{QCD}, how do the interaction parameters of the effective theory depend on it? Finding the answer to this question is, of course, a very ambitious enterprise which will not be resolved in this paper.

One notable attempt, related to the approach of this paper, is the work of Lüscher, van Baal and collaborators [8] who use a "non-perturbative" approximation of Born-Oppenheimer type wherein one tries to integrate out "fast" (short wavelength) fluctuations thereby leaving an effective theory for the "slow" (long wavelength) modes. In the case of QCD if this separation is carried out considering the theory to be confined in a small box with periodic boundary conditions, which acts as an infrared cutoff, then the fast modes can be integrated out using perturbation theory. In fact, as the QCD vacuum is translationally invariant, we can consider integrating out all non-constant modes thereby yielding an effective field theory of constant gauge fields. In effect, one is now considering a one-dimensional field theory, i.e. quantum mechanics. In the case of the pure gauge theory the spectrum of the resulting Hamiltonian will lead to the glueball spectrum. However, this is the glueball spectrum associated with a "universe" that is smaller than a fermi. To describe the real world it is necessary to take a box that has at least a few times the size of the glueball. Unfortunately, this leads to a breakdown in the perturbation theory that is used to integrate out the non-constant modes. The physical reason is simple: the perturbation theory is associated with free gluons, however, the long wavelength modes are better described by other non-perturbative effective degrees of freedom. Thus, although the methodology in principle is a very powerful one, leading to an effective field theory that depends only on the running coupling constant, in practice one finds that in trying to extend it to larger boxes one encounters the same fundamental problem that led to the introduction of the box in the first place.

[a] On sabbatical leave from the Instituto de Ciencias Nucleares, UNAM, Mexico;
e-mail: hess@theo.physik.uni-giessen.de
[b] e-mail addresses: alerma@nuclecu.unam.mx, vieyra@pythia.nuclecu.unam.mx stephens@nuclecu.unam.mx, axel@nuclecu.unam.mx

Here we will take a somewhat less ambitious path motivating an ansatz for an effective Hamiltonian via an analysis of the group-theoretical structure of QCD, then analyzing some of the consequences of this ansatz, in particular the spectrum of glueball masses. First of all we will take the effective Hamiltonian to depend on spatially constant modes only. From the point of view of the functional integral in principle it is always possible to integrate out the non-constant modes. However, if one is implementing an approximation technique to do it one has to check that the resulting approximation is trustworthy. As mentioned this has been investigated by Lüscher, van Baal and collaborators [8]. Their approximation breaks down when long wavelength gauge modes (other than the zero mode) become strongly coupled. Nevertheless, given the translation invariance of the QCD vacuum one would still expect an expansion around the ground state to be an expansion around a spatially constant state. We will take that attitude here and work with an effective Hamiltonian that only depends on spatially constant modes. We will restrict attention to the pure gauge sector of QCD. Hence, when considering excited states we will only be considering glueballs. Given the present experimental status, it is difficult to compare our results to the outcome of actual measurements, however, we can compare with results obtained from lattice simulations that restrict to the pure gauge sector.

As a first step, in Sect. 2, we formulate a classification scheme for a many-gluon system, assuming the dominance of only one mode with $J^P = 1^-$. This will enable us to order many-gluon states into bands defined by the irreducible representations of an $O(8)$ group. This step alone plays an important rôle, allowing us to explain the level ordering observed in [9,10]. This result is to a large extent model independent, the only assumption being that gluon number (here we are referring to constituent gluons not perturbative $g \to 0$ gluons) is approximately a good quantum number, and that the more gluons in a state the higher the corresponding energy.

Next, in Sect. 3, we make an ansatz for the effective Hamiltonian of QCD in terms of spatially constant gauge fields motivating the use of a kinetic energy that is quadratic in momenta. We further introduce a change of variables on the space of spatial gauge potentials such that the new variables are related to the intensity of the spatial gauge field, the spatial quadrupole distribution and the rotational angles in space and colour space. These variables have been widely used in nuclear physics [11,12] and also in QCD [8,13]. Though we work with constant modes here, this restriction is not necessary in general. In fact, the new variables as functions of space and time qualify as candidates for effective fields that better describe low energy QCD. With a simple ansatz for the potential term the resulting effective Hamiltonian turns out to be a function of a one-dimensional oscillator, a five-dimensional oscillator and some Casimir operators of symmetry groups appearing in the problem of many gluon systems. The first oscillator describes changes in the intensity of the gluonic field while the second is related to the quadrupole deformation of the gluonic field with respect to an "intrinsic" system yet to be defined (it will be similar in structure to the "body fixed frame" of a nucleus). One feature of the effective Hamiltonian is that there exists a conserved quantity which one can naturally identify with the number of constituent gluons.

The resulting effective Hamiltonian depends on several parameters which we determine in Sect. 4 by fitting the corresponding spectrum to several glueball states with spin, parity and charge conjugation J^{PC} and masses known from lattice gauge calculations. We subsequently make predictions for all other states and compare, where possible, with lattice simulations and results from other phenomenological models, such as the MIT bag model. Finally in Sect. 5 we draw our conclusions.

2 Gluon taxonomy

As a first step to understanding the structure of the spectrum of a many gluon system one has to determine which values of J^{PC} can appear, where J refers to spin and P, C to parity and charge conjugation, respectively. It is implicitly understood that only colour zero states are allowed. To deal with this problem in an efficient way it is not sufficient to consider the $SU(3)$ colour and $SO(3)$ spin groups only.[1] One reason for this is that gluon fields do not transform according to the fundamental irreducible representation (irrep) of the colour group. The main point, however, is that we are dealing with a system of identical bosons, hence it is necessary to implement Bose-Einstein statistics. This can be done most conveniently by considering higher groups. With respect to colour rotations, the gluonic fields transform like vectors in an eight-dimensional space so we can associate a rotation group $O(8)$ with colour transformations. Extending the latter rotations to unitary transformations, we arrive at the group $U(8)$ [14]. In the same way the spatial rotations can be associated with an $U(3)$ group containing the $SO(3)$ spin group. Bose-Einstein statistics are then implemented by connecting at this level the spin and colour transformations formally to an $U(24)$ group.

We thus obtain the following group chain [15]:

$$
\begin{array}{ccccccc}
[N] & & & & [h_1h_2h_3] & & \\
\boldsymbol{U(24)} & \supset & \boldsymbol{U(8)} & \times & \boldsymbol{U(3)} & & \\
 & & \xi \ \cup & & \cup & & \\
 & & (\omega_1\omega_2\omega_30)\ \boldsymbol{O(8)} & & \boldsymbol{SU(3)_J} & & (p,q) \\
 & & \cup & & \cup & & K \\
 & & (0,0)\ \boldsymbol{SU(3)} & & \boldsymbol{SO(3)} & & J,M
\end{array}
\tag{1}
$$

where for each group the quantum numbers of the corresponding irrep are given and ξ and K are multiplicity indices appearing in the respective reductions. The symbol $[N]$ denotes the completely symmetric irrep of $U(24)$ for N

[1] Note that we use $SO(3)$ rather than $SU(2)$ as we are restricting our attention to integer spins

gluonic modes, which implements the Bose-Einstein statistics. The possible irreps of $U(8)$ and $U(3)$ are then given by the Young tableaux $[h_1h_2h_3]$, consisting of three rows with h_1, h_2 and h_3 boxes, respectively, where $h_1+h_2+h_3=N$. The fact that the $U(8)$ and $U(3)$ irreps are coupled to the completely symmetric irrep of $U(24)$ constrains their Young tableaux to be equal [14].

The $U(3)$ group reduces via $SU(3)_J$, where J denotes spin, to the rotation group $SO(3)$. The reduction rules are given in [16], but will be summarized here for easier reference. First the reduction from the irrep $[h_1h_2h_3]$ of $U(3)$ to (p,q) of $SU(3)_J$ is simply given by $p=h_1-h_2$ and $q=h_2-h_3$. The possible irreps J (with magnetic quantum number M) of $SO(3)$ are then determined by the possible values of the multiplicity index K,

$$K=\min(p,q),\min(p,q)-2,\ldots,0 \text{ or } 1\ , \qquad (2)$$

and

$$J=\max(p,q),\max(p,q)-2,\ldots,0 \text{ or } 1 \text{ for } K=0$$
$$J=K,\ K+1,\ldots,K+\max(p,q) \text{ for } K\neq 0\ . \qquad (3)$$

The group that describes rotations in colour space is $U(8)$, and it reduces via $O(8)$ to the physical colour group $SU(3)$, where we restrict to colour singlets, i.e. to the trivial irrep $(0,0)$. The intermediary orthogonal group $O(8)$ will play a very important rôle in the following developments. The $O(8)$ quantum numbers are $(\omega_1\omega_2\omega_30)$, where only three of the four numbers can be non-zero as a consequence of the restriction of the $U(8)$ irrep to three rows [17]. The largest $O(8)$ irrep contained in a given irrep $[h_1h_2h_3]$ of $U(8)$ is the one with $\omega_k=h_k$, $k=1,2,3$. The precise rules for the reduction of $U(8)$ to $O(8)$ are given, for instance, in [17] (there is also a computer program available for arbitrary irreps of different classical groups [18]). The reduction of $O(8)$ to $SU(3)$ will be carried out in a recursive manner using the reduction of $U(8)$ to $O(8)$ just mentioned and the reduction of $U(8)$ to $SU(3)$. The latter is given explicitly for up to five (constituent) gluons in [15], and the reduction for an arbitary irrep of $U(8)$ can be found in [19]. The recursive procedure for the reduction of $O(8)$ to $SU(3)$ is described in detail in Appendix A. The results are given for up to six constituent gluons in Table 1, considering only those irreps of $U(8)$ and $O(8)$ which contain at least one colour zero irrep. Finally, note that the multiplicities appearing in the reduction of $U(8)$ to $O(8)$ and of $O(8)$ to $SU(3)$ are denoted globally by ξ in (1).

The above classification is not applicable in its present form to cases where more than one gluonic mode is relevant. It is, however, always valid when one particular – even non-constant – mode is considered to be dominating at low energies, subject only to the condition that it has spin 1, colour $(1,1)$ and, in view of our further considerations, parity (-1).

In order to discuss the properties of the states under parity and charge conjugation we introduce boson creation and annihilation operators, $b_a^{i\dagger}$ and b_a^i, for the gluon field A_a^i, i.e. we write $A_a^i\sim(b_a^{i\dagger}+b_a^i)$ with a proportionality factor depending on certain parameters. What we will present in this section is independent of these parameters, which will be fixed in the next section. The operators $b_a^{i\dagger}$ and b_a^i satisfy the usual commutation relations.

Table 1. Reduction of $U(8)$ to $O(8)$ up to six constituent gluons taking into account only those $U(8)$ irreps which contain at least one colour scalar (the actual number of colour scalars is indicated in the third column). Also the reduction of $U(3)$ via $SU_J(3)$ to the angular momentum group $SO(3)$ is given, and we have indicated the corresponding values of J. Subindices refer to multiplicities which can be distinguished by the multiplicity index K. P and C refer to parity and charge conjugation, respectively

$U(8)$ $(U(3))$	$O(8)$	#(0,0)	SO(3) (J)	P	C
[2]	(0000)	1	0,2	+1	+1
[4]	(0000)	1	0,2,4	+1	+1
$[2^2]$	(0000)	1	0,2	+1	+1
[6]	(0000)	1	0,2,4,6	+1	+1
[42]	(0000)	1	$0,2_2,3,4$	+1	+1
$[2^3]$	(0000)	1	0	+1	+1
[3]	(3000)	1	1,3	−1	−1
[5]	(3000)	1	1,3,5	−1	−1
[41]	(3000)	1	1,2,3,4	−1	−1
[32]	(3000)	1	1,2,3	−1	−1
$[1^3]$	(1110)	1	0	−1	+1
$[31^2]$	(1110)	1	0,2	−1	+1
$[21^2]$	(2110)	1	1	+1	−1
$[41^2]$	(2110)	1	1,3	+1	−1
[321]	(2110)	1	1,2	+1	−1
$[2^2]$	(2200)	1	0,2	+1	+1
[42]	(2200)	1	$0,2_2,3,4$	+1	+1
[321]	(2200)	1	1,2	+1	+1
$[2^3]$	(2200)	1	0	+1	+1
$[2^21]$	(2210)	1	1	−1	−1
$[31^2]$	(3110)	1	0,2	−1	−1
[6]	(6000)	1	0,2,4,6	+1	+1
[42]	(4200)	1	$0,2_2,3,4$	+1	+1
$[41^2]$	(4110)	1	1,3	+1	−1
$[3^2]$	(3300)	1	1,3	+1	−1
$[2^3]$	(2220)	1	0	+1	+1

A basis of states can be characterized unambiguously by the quantum numbers in (1) (some of them are redundant, see also below). Any such state can be obtained by applying the pair operators

$$q_M^{[J]\dagger}=\sum_a [b_a^\dagger\times b_a^\dagger]_M^{[J]} \qquad (4)$$

to a minimum weight state, i.e. a state with $h_k=\omega_k$, $k=1,2,3$. The notation $[A\times B]_M^{[J]}$ in (4) represents the spin coupling of two tensors A and B to spin J [20], where in our case $J=0,2$ due to the spin and the bosonic nature of the gluons. Note also that the pair operators $q_M^{[J]\dagger}$ are $O(8)$

scalars because of the summation over a. The conjugate operators $q_M^{[J]}$ annihilate the minimum weight states.[2]

A general basis state can then be decomposed as follows:

$$\begin{aligned}\Big|[h_1h_2h_3]\,\xi\,(\omega_1\omega_2\omega_30), KJM\Big\rangle \\ = \Big[\mathcal{P}(q^\dagger)^{[2n_1,2n_2,2n_3]} \\ \otimes \Big|[\omega_1\omega_2\omega_3]\,\xi'\,(\omega_1\omega_2\omega_30)\Big\rangle\Big]_M^{[h_1h_2h_3]KJ}\,. \end{aligned} \quad (5)$$

The symbol $\mathcal{P}(q^\dagger)_{M_1}^{[2n_1,2n_2,2n_3]K_1J_1}$ represents a coupling of $(n_1+n_2+n_3)$ gluon pair operators from (4) to the $U(8)$ (or $U(3)$) irrep $[2n_1,2n_2,2n_3]$ and to spin J_1 with multiplicity index K_1. Due to the commutativity of the pair operators the Young diagram $[2n_1,2n_2,2n_3]$ must have an even number of boxes in each row [17]. The operator $\mathcal{P}(q^\dagger)$ is subsequently coupled with a minimum weight state to the total $U(8)$ (or $U(3)$) irrep $[h_1h_2h_3]$ and to spin J, the $U(3)$ (or equivalently $U(8)$) coupling being indicated by $\otimes$. This coupling is such that

$$N = h_1+h_2+h_3 = 2(n_1+n_2+n_3)+\omega_1+\omega_2+\omega_3\,. \quad (6)$$

It is natural to interpret the quantum number N as the number of constituent gluons. One should, however, be careful not to identify N with the number of the constituent gluons which arise in other phemonenological models, as discussed in Sect. 4. At any rate, (6) gives the precise definition of N in our model.

We did not explicitly indicate in (5) the quantum numbers N and (p,q), which are uniquely determined by h_1, h_2, h_3, the latter via $p=h_1-h_2$ and $q=h_2-h_3$. Furthermore it is understood that the state in (5) is a singlet with respect to the colour $SU(3)$.[3] The multiplicity label ξ is replaced by ξ' in the second representation in (5) because part of the multiplicity is taken care of by the different couplings of $[2n_1,2n_2,2n_3]$ with $[\omega_1\omega_2\omega_3]$ to $[h_1h_2h_3]$. The division of the state into a coupling of $(n_1+n_2+n_3)$ gluon pairs and the "rest" is related to the concept of generalized seniority, where a particle system can be divided into pairs of particles coupled to colour zero and spin 0 or 2 and the rest where no pairs of these types appear. The term "generalized" is used because seniority is normally associated with a coupling to a total scalar.

The key point of the decomposition (5) is that the gluon pair operators change neither parity nor charge conjugation, which are therefore already determined by the minimum weight states, i.e. the ones with $[h_1h_2h_3] = [\omega_1\omega_2\omega_3]$. Furthermore, it turns out that the minimum weight states carry definite parity and charge conjugation, as will be demonstrated in the following. It is due to these facts that the $O(8)$ irreps play a prominent rôle in the classification of many-gluon states.

The parity of an N-gluon state is readily determined by noting that each gluon field A_a^i – and therefore each operator $b_a^{i\dagger}$ – carries parity (-1). Therefore, the parity of a state with $O(8)$ irrep $(\omega_1\omega_2\omega_30)$ is given by

$$P = (-1)^N = (-1)^{\omega_1+\omega_2+\omega_3}\,. \quad (7)$$

Obtaining the charge conjugation of the states is considerably more involved. The reason can be seen by considering the simple example of a one-gluon state. The properties of A_a^i (and hence of $b_a^{i\dagger}$) under charge conjugation are deduced in [21] and turn out to be

$$CA_a^i = \eta_a A_a^i \quad (\text{no sum over } a)\,, \quad (8)$$

with

$$\eta_a = \begin{cases} -1 \text{ for } a = 1,3,4,6,8 \\ +1 \text{ for } a = 2,5,7 \end{cases} \quad (9)$$

in the standard representation of $SU(3)$, where the generators are proportional to the Gell-Mann matrices. Since one-gluon states belong to the $U(8)$ and $O(8)$ irreps [1] and (1000), respectively, we have an example of an $O(8)$ irrep lacking a definite charge conjugation. As we will show in the sequel one can, contrary to this observation for one-gluon states, associate definite charge conjugations with all colour-zero states.

In order to see that, we have to construct the states from so-called elementary permissible diagrams (epds) [22]. An epd is, in our case, a minimum weight state as considered before, coupled to colour zero and definite spin, which is elementary in the sense that it cannot be decomposed into other couplings. It will be denoted by

$$(N,[\omega_1\omega_2\omega_3],J) = [b^\dagger\otimes\cdots\otimes b^\dagger]_{M=J}^{[\omega_1\omega_2\omega_3]J}\,, \quad (10)$$

where $N=\omega_1+\omega_2+\omega_3$ and the $O(8)$ irrep is $(\omega_1\omega_2\omega_30)$. Note that we always couple to $M=J$, which implies that a product of two epds, say $(N,[h_1h_2h_3],J)(N',[h_1'h_2'h_3'],J')$, represents a state with spin $J+J'$. However, such a product of epds has still to be projected onto a definite $U(3)$ (or equivalently $U(8)$) irrep, meaning that the epds are defined here with respect to the maximum weight of $SO(3)$ and not of $U(3)$ ($U(8)$).

Any minimum weight state can be represented by products of all possible epds. Including the pair operators of (4) (with $M=J$) in the list of epds, we can then via (5) construct any state with the quantum numbers of (1) from the coupling of epds.[4] There exist systematic procedures to determine the total number of epds, and the application of these general methods to the case at hand (including states with colour which are important in

[2] Together with the generators of the $U(8)$ group these operators form the algebra of the symplectic group $Sp(6,R)$, which, however, will not be considered further here (see [12] and references therein)

[3] In the pure gauge theory, the gluons have to be coupled to colour $(0,0)$, which is not necessarily true in the presence of quarks. The notation in (5) can be readily extended to the latter case

[4] As a detailed example for the construction of states with the help of epds see the discussion of the five-dimensional harmonic oscillator in [23]

the presence of quarks) is presently worked out [24]. However, for the time being, we will adopt a more pedestrian approach which is completely satisfactory in the case of small gluon numbers. It should be noticed that in general the choice of the set of epds is not unique being somewhat analogous to the choice of a basis for a vector space.

The epds with up to three constituent gluons are (summation over repeated indices is understood)

$$\begin{aligned}(2,[2],J_1) &= [b_a^\dagger \times b_a^\dagger]^{[J_1]}_{J_1} \\ (3,[3],J_2) &= d_{abc}\left[[b_a^\dagger \times b_b^\dagger]^{[2]} \times b_c^\dagger\right]^{[J_2]}_{J_2} \\ (3,[1^3],J_3) &= f_{abc}\left[[b_a^\dagger \times b_b^\dagger]^{[1]} \times b_c^\dagger\right]^{[J_3]}_{J_3}\,, \end{aligned} \tag{11}$$

the first one being the pair operator (4), while the others are minimum weight states in the sense defined before. The spins denoted by J_k are limited to the values $J_1 = 0, 2$, $J_2 = 1, 3$ and $J_3 = 0$. A complete list of epds with up to six gluons, as well as the decomposition of some minimum weight states with seven or eight gluons to be used in the discussion of the glueball spectrum, can be found in Appendix B.

We can now use the decomposition of the states into epds to determine their charge conjugation, using the fact that the epds are charge conjugation eigenstates. For example, by construction the epd of two gluons in (11) has charge conjugation $C = +1$. For the epds with three gluons we have for the symmetric coupling $C = -1$ and for the antisymmetric coupling $C = +1$, which can be readily verified taking into acount the combinations of (a, b, c) which appear in these couplings and using the list in (9). In general, as shown in Appendix C, the total charge conjugation is simply given by

$$C = (-1)^{n_d}\,, \tag{12}$$

where n_d is the number of d-symbols appearing in the coupling. Using this fact and the list of epds in Appendix B we can deduce the charge conjugation of all states constructed via (5) from minimum weight states with up to six gluons. The results are given in Table 1.

3 An effective hamiltonian for QCD

In this section we will motivate an ansatz for an effective Hamiltonian for purely gluonic QCD at low energies. The Hamilton density in the temporal gauge is of the form

$$\mathcal{H} = -\frac{1}{2}\sum_{i,a}(\partial_0 A_a^i)^2 + \mathcal{V}(A)\,, \tag{13}$$

where the first term is the kinetic energy density and the second is the potential which includes everthing else, such as $(\nabla \times \mathbf{A}_a)^2$.

To obtain the effective action of the quantum theory one must integrate over all possible physically inequivalent configurations of the gauge field. This is, except in certain special cases, impossible to do exactly. The most popular approximation technique is perturbation theory, however, for a discussion of QCD at low energies this simply will not do. Another possible approximation, as mentioned previously, is of Born-Oppenheimer type [8] wherein one tries to integrate out "fast" (short wavelength) fluctuations thereby leaving an effective theory for the "slow" (long wavelength) modes. Here, we make an ansatz for the low-energy effective Hamiltonian, taking it as a function of spatially constant gauge fields only. Given the translation invariance of the QCD vacuum, this is not physically unreasonable. The basic assumption behind this ansatz is that the dynamics of the constant modes alone can give us information about the glueball spectrum, i.e. the excited states of the theory, although obviously the inner structure of the states cannot be described by constant modes. As was shown in the work of Lüscher, van Baal and Koller [8], this assumption is correct for the system located inside a box with periodic boundary conditions and a size of up to 0.7 fm (approximately).

After having integrated over all non-constant modes our general ansatz for the quantized effective Hamiltonian is

$$H_{eff} = -\frac{1}{2B_\varrho}\sum_{i,a}\frac{\partial^2}{(\partial A_a^i)^2} + V_{eff}(A)\,, \tag{14}$$

where the effective potential V_{eff} is a complicated function of A_a^i that will be modelled by a simple ansatz further below. The approximation effected in (14) consists in neglecting higher derivatives with respect to A, as well as terms mixing derivatives with powers of A. This adiabatic approximation has been justified in the case of a finite volume and to one-loop order [8], and here we will simply assume that for our purposes the same approximation can also be applied to the more general case. The constant B_ϱ is a wave function renormalization, which we will treat as an adjustable parameter.

After choosing the temporal gauge and integrating out all non-constant modes there is still a residual gauge symmetry left in (14), namely the invariance under spatially constant, i.e. global (and time-independent) gauge transformations. We hence impose on the physical states the condition

$$G_a \Psi_{phys}(A) = 0\,, \tag{15}$$

where the G_a are the generators of global gauge transformations. Equation (15) is equivalent to the statement that physical states have no colour, in which form the condition was imposed on the many-gluon states considered in the previous section. The consistency of the present gauge fixing procedure with the integration over all non-constant modes was shown in [8].

We will now change variables. The transformation is chosen such that part of the new variables is intimately related to the gluon pairs introduced in the previous section. The interpretation of these variables will be given below. Explicitly the transformation is given by [12]

$$A_a^i = \sum_{k=1}^{3} \varrho_k D^{1_3}_{ki}(\theta)\Delta^{1_8}_{5+k,a}(\phi)\,. \tag{16}$$

where the constant modes of the vector potential A_a^i depend on the spatial indices ($i = 1,2,3$) and the colour indices ($a = 1,2,...,8$). $D_{ki}^{1_3}$ is the rotation matrix in the spatial three-dimensional space with cartesian components and $\Delta_{5+k,a}^{1_8}$ is the corresponding rotation matrix in the colour space of dimension 8. Of the latter only the last three rows appear. The angles ϕ_{ab} for rotations in colour space can be chosen in such a way that only 18 angles appear in (16). Together with the three angles θ_{ij} for spatial rotations and the three ϱ_k we then have (formally) 24 degrees of freedom as is the case for the A_a^i on the l.h.s. of (16). Eight of the angles ϕ_{ab} correspond to global gauge transformations, hence the generators G_a in (15) can be identified with the angular momentum operators for the corresponding rotations. The condition (15) then simply shows that physical states are independent of these eight angles, thereby effectively reducing the number of degrees of freedom to 2×8, as might have been anticipated.

Transformations similar to (16) have been used in other areas of physics [11,12] and also in the work of van Baal and Koller [8]. In both cases the coordinates were taken to be space independent as in (16). However, (16) can be readily extended to the more general case where the coordinates include space (and time) dependence. In any case, the interpretation of this coordinate transformation given below is the same irrespective of whether the coordinates are space dependent or not.

We will now consider the interpretation of these new variables. We first define the composite field

$$q_{ij} = \sum_a A_a^i A_a^j = \sum_k D_{ki}^{1_3}(\theta) \varrho_k^2 D_{kj}^{1_3}(\theta) \,. \tag{17}$$

Equation (17) gives the rotation in three-dimensional space of the matrix (q_{ij}) into a system in which the matrix is diagonal, which will be called the "intrinsic system". The matrix (q_{ij}) has both monopole and quadrupole components which are obtained by simply coupling to the appropriate spin, i.e.

$$q_M^{[J]} = \sum_{i,j} (1i, 1j \mid JM)\, q_{ij} \,, \tag{18}$$

where $J = 0, 2$ indicates the spin and $(1i, 1j \mid JM)$ is a Clebsch-Gordan coefficient coupling two cartesian vectors to a spherical tensor of spin J. Equation (18) represents of course the analogue to the gluon pair operators $q_M^{[J]\dagger}$ of the previous section (see (4)). For $J = 0$,

$$q_0^{[0]} = \frac{1}{\sqrt{3}} \sum_{i,a} (A_a^i)^2 = \frac{1}{\sqrt{3}} \sum_k \varrho_k^2 \equiv \frac{\varrho^2}{\sqrt{3}} \tag{19}$$

gives the square of the intensity of the vector field. The spin 2 part describes the quadrupole distribution of the intensity of the vector field. Thus the ϱ_k carry the information about the monopole and quadrupole distribution. In (16) and (17) the angles θ_{ij} give the orientation of the intrinsic system with respect to the laboratory system and can be defined in many different ways. For instance, instead of the rotation angles θ_{ij} in two-dimensional planes one could just use Euler angles. The angles ϕ_{ab} describe analogous rotations in colour space.

The monopole-quadrupole tensor (q_{ij}), being quadratic in the gauge field, can only describe excitations with an even number of gluons. In addition, by the summation over colour indices in (17), the variables and thus the excitations they describe do not change parity nor charge conjugation. Of course, "non-pair" excitations can still be described via (16) and will be classified by $O(8)$ irreps.

In summary, our conjecture is that the variables ϱ_k, ϕ_{ab} and θ_{ij} represent effective degrees of freedom that will be useful in describing the collective behaviour of the gluon field, in the same way as in nuclear physics analogous variables describe the collective rotational and vibrational modes of the nucleus. We believe that a generalization of these variables to include space-time dependence could also be useful in QCD.

As described in [12], instead of the variables ϱ_k one can use ϱ, b and c which are related to the former by

$$\varrho_k^2 = \frac{\varrho^2}{3} \left[1 + 2b \cos\left(c - \frac{2\pi}{3} k \right) \right] \,, \tag{20}$$

where $k = 1,2,3$. The parameter b gives the absolute deformation and c its deviation from axial symmetry. There are restrictions however [12,25]: Firstly, the sector from $c = 0$ to $\pi/3$ already represents all possible physically different situations. The other sectors (six in total) can be mapped bijectively to the one described [25]. The Jacobian of the transformation (20) becomes singular on the boundaries between different sectors. Secondly, the restriction that the left-hand side of (20) be positive imposes a c-dependent upper limit upon the variable b.

It will turn out to be even more convenient to replace b by the variable β related to it via [12]

$$1 + 2\left(\frac{\beta}{\sigma}\right)^2 = \frac{1 - b^2}{1 - 3b^2 + 2b^3 \cos(3c)} \tag{21}$$

with the scale factor σ defined in (22) below. The range of β is from 0 to ∞ (see [25]). For b or (β/σ) small we have $\beta/\sigma \approx b$.

Now we will rewrite the Hamiltonian of (14) in terms of the new variables of (16)–(21). For the time being we will concentrate on the kinetic term $T = H_{eff} - V_{eff}$. In effect, for this case the transformation to the new variables has already been performed in [12] where the number A, referring to the number of nucleons, has to be replaced by 9 and in particular the orthogonal group $O(A-1)$ becomes $O(8)$, otherwise nothing changes.

We will in the following briefly outline the procedure and give the main definitions and results. In the first step the kinetic term T is rewritten in terms of the variables ϱ_k ($k = 1,2,3$) and the components of the spin operator $\mathbf{J}_k$ in the intrinsic system associated with the angles θ_{ij}. T also depends on the generators $\mathcal{L}_{5+k_1,5+k_2}$ ($k_1, k_2 = 1,2,3$)

which form an $SO(3)$ subgroup of $O(8)$. As shown in [12] and [26], the matrix elements of the generators approach for large

$$\sigma^2 = \omega_1 + \omega_2 + \omega_3 + 6 \tag{22}$$

the matrix elements of the generators $\mathcal{L}_k$ $(k = 1, 2, 3)$ of the $SO(3)'$ subgroup of an $U(3)'$ group in the irrep $[\omega_1\omega_2\omega_3]$. The primes indicate that a contraction procedure [27] was used in terms of the expansion parameter $(1/\sigma^2)$. The $[\omega_1\omega_2\omega_3]$ irrep of $U(3)'$ contains the irrep $(\lambda = \omega_1 - \omega_2, \mu = \omega_2 - \omega_3)$ of its $SU(3)'$ subgroup, and the latter can be reduced to $SO(3)'$ by using the rules given in (2) and (3). We will identify the components of $\mathcal{L}$ with angular momentum operators later on.

The volume element in terms of the new variables is

$$(\varrho_1^2 - \varrho_2^2)(\varrho_1^2 - \varrho_3^2)(\varrho_2^2 - \varrho_3^2)(\varrho_1\varrho_2\varrho_3)^5 d\varrho_1 d\varrho_2 d\varrho_3 d\Omega_\theta d\Omega_\phi \,, \tag{23}$$

where $d\Omega_\theta$ and $d\Omega_\phi$ refer to the volume element of the respective angles. Their explicit form depends on the specific angles chosen (Euler or others). We have also chosen $\varrho_1 \geq \varrho_2 \geq \varrho_3$ corresponding to the sector in the (b, c)-plane with $0 \leq c \leq \pi/3$ in order to avoid ambiguities in the choice of the intrinsic system.

Next, we change the volume element via the transformation

$$\Phi = (\varrho_1\varrho_2\varrho_3)^{5/2}(\varrho_1^2 + \varrho_2^2 + \varrho_3^2)^2 \Psi \tag{24}$$

to the new states Φ in the Schrödinger picture. The change of volume element is such that for small $\beta/\sigma \approx b$ the new volume element takes the form

$$\left(\frac{\beta}{\sigma}\right)^4 \sin(3c)\, d\varrho\, d\beta\, dc\, d\Omega_\theta\, d\Omega_\phi \tag{25}$$

as given in [25], neglecting corrections to the next order in (β/σ).

Implementing these changes, the kinetic term in the effective Hamiltonian subsequently acquires the form

$$T = \frac{1}{2B_\varrho}\left(-\frac{\partial^2}{\partial\varrho^2} + \frac{\mathbf{R^2}}{\varrho^2}\right) , \tag{26}$$

where $\mathbf{R}^2$ is an operator which contains linear and quadratic derivatives with respect to the variables β and c. It is also a function of the spin operators $\mathbf{J}_k$ in the intrinsic system and of the last three generators of the $O(8)$ group, replaced by $\mathcal{L}_k$ after the contraction discussed above. Additionally, it depends on combinations of Casimir operators of the $O(8)$ group and of its subgroups $O(7)$, $O(6)$ and $O(5)$ in the canonical chain.

In [12] an expansion is made in powers of the parameter $(1/\sigma^2)$ defined in (22). In the case of [12] this expansion parameter is always very small. In the case at hand, where it starts from 1/6 and decreases with larger ω_k $(k = 1, 2, 3)$, it is not too small, however one would expect the expansion not to be too bad. Actually, we will expand not only in powers of $(1/\sigma^2)$ but also in powers of (β/σ), leading to the contributions summarized by the formulas (5.6) to (5.11) of [12]. However, it is necessary to keep the term given by (5.11) in this reference ($\mathbf{R}_2^2$ below) because it can be, in our case, of the same order as σ^2. We thus arrive at the following expression for the $\mathbf{R}^2$ operator of (26):

$$\mathbf{R}^2 = \mathbf{R}_1^2 + \mathbf{R}_2^2 \,, \tag{27}$$

where

$$\begin{aligned}\mathbf{R}_1^2 \approx 2\sigma^2 \Bigg[&- \frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{1}{\beta^2\sin(3c)}\frac{\partial}{\partial c}\sin(3c)\frac{\partial}{\partial c} \\ &+ \sum_{k=1}^{3}\frac{\mathbf{L}_k^2}{2\beta^2\sin^2(c - 2\pi k/3)}\Bigg] \\ &+ \left(\sigma^4 - \frac{9}{4}\right)\left(1 + 2\left(\frac{\beta}{\sigma}\right)^2\right) + 12\end{aligned} \tag{28}$$

and

$$\mathbf{R}_2^2 \;\approx\; 2\,\mathcal{C}_2(\lambda,\mu) - 3\sum_{k=1}^{3}\mathcal{L}_k^2 \tag{29}$$

with

$$\begin{aligned}\mathbf{L}_k &= \mathbf{J}_k + \mathcal{L}_k \\ \mathcal{C}_2(\lambda,\mu) &= \lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu) \\ \lambda &= \omega_1 - \omega_2, \quad \mu \;=\; \omega_2 - \omega_3 \,,\end{aligned} \tag{30}$$

the ω_k being the quantum numbers of the $O(8)$ irrep.

The above is a good approximation in the case that (β/σ) is very small. However, as this is not entirely the case we will modify the expression such that the contributions from higher orders in (β/σ) are taken into account via a redefinition of the interaction parameters (a procedure commonly used in the collective model of nuclei [25]). At this point our model begins to look even more phenomenological, similar to a Landau-Ginzburg ansatz, where the theory contains parameters which have to be adjusted to some kind of "experiment", in our case to lattice calculations. Nevertheless, the kinematical structure is maintained with the hope of learning something about the spectral structure of pure gluonic QCD.

In order to illustrate the parameter redefinition we consider the term in square brackets in (28). It can immediately be identified with the kinetic energy associated with a quadrupole degree of freedom as given in [25], where the spin operators $\mathbf{J}_k$ are replaced by $\mathbf{L}_k = \mathbf{J}_k + \mathcal{L}_k$.

In principle, the full kinetic term (26) (after the reduction in terms of the parameter $(1/\sigma^2)$) can be written down: it turns out still to be quadratic in the derivatives, with the coefficients given as power series in (β/σ). This is extensively discussed in [25]. It is straightforward but cumbersome to obtain the corresponding coefficients of these higher terms in the kinetic energy. A much simpler procedure takes the higher contributions into account by redefining the coefficients of the lowest-order terms. In the example considered above, we would multiply the square bracket in (28) by a parameter $(1/B_2)$, to be determined "experimentally". This procedure is sensible as long as the expansion parameter (β/σ) does not vary over a wide range [25].

If we add to the terms in square brackets in (28) a contribution proportional to β^2, we obtain the Hamiltonian of a five-dimensional harmonic oscillator. In fact, such a term is available in (28), and we multiply it – following the above philosophy – by a factor C_2 (there is another term relatively suppressed by a factor $(1/\sigma^4)$, which we neglect). We can then write the whole expression as the number operator $\mathbf{N}_2$ of the quadrupole bosons times a constant depending on (C_2/B_2), plus the zero-point energy.

Taking into account all the terms in (28) and (29), the operator $\mathbf{R}^2$ is given by a linear combination of the harmonic oscillator of the quadrupole degree of freedom discussed above, the eigenvalue $\mathcal{C}_2$ of the Casimir operator of the $SU(3)'$ group, the $\mathcal{L}^2$ operator of the $SO(3)'$ group contained in $SU(3)'$, a term proportional to σ^4 and a constant. Therefore, as an *ansatz* for $\mathbf{R}^2$ we propose

$$\mathbf{R}^2 = 4\sqrt{\frac{C_2}{B_2}}\sigma^2\mathbf{N}_2 + k_1(\sigma^4 - 36) + k_2\mathcal{C}_2(\lambda,\mu) + k_3\sum_{k=1}^{3}\mathcal{L}_k^2 + k_4 - \frac{1}{4}\,. \tag{31}$$

For convenience we have absorbed the zero-point energy of the harmonic oscillator and a term $(36k_1 + 1/4)$, in the parameter k_4 multiplying the constant term.

Now we turn to the potential term V_{eff} in (14). In general, V_{eff} will be a complicated function of the three fundamental tensors $I_1 = \varrho^2$, $I_2 = \sqrt{1/5}\,\beta^2$ and $I_3 = -\sqrt{2/35}\,\beta^3\cos(3c)$ (see [25]), and also of those angles ϕ_{ab} which are not excluded by gauge invariance. In keeping with the expansion in powers of (β/σ), we approximate V_{eff} by a function of ϱ^2 only. We also neglect any dependence on the angles ϕ_{ab} which has to be considered as an *ad hoc* ansatz in the present context. There is, however, some motivation for this omission from the perturbative expressions calculated in [8]. We will further approximate the potential in the vicinity of its minimum by a linear combination of the terms $1/\varrho^2$ and ϱ^2, where the former can be assimilated into $\mathbf{R}^2$ via a redefinition of the constant k_4 in (31). In fact, all the contributions to V_{eff} which happen to have the same structure as one of the terms in $\mathbf{R}^2$, can be taken into account by redefining the corresponding parameters. The ϱ^2 term in the potential can be interpreted as a gluon mass term as is clear from (19). We remark that, despite this mass term, the number of degrees of freedom of the spatially constant gluon field corresponds to transverse gluons due to the remnant of Gauss' law (15).

We then arrive at the final form of our ansatz for the effective Hamiltonian,

$$H_{eff} = -\frac{1}{2B_\varrho}\frac{\partial^2}{\partial\varrho^2} + \frac{\mathbf{R}^2}{2B_\varrho\varrho^2} + \frac{C_\varrho}{2}\varrho^2\,, \tag{32}$$

with $\mathbf{R}^2$ as in (31) and a new parameter C_ϱ from the modelling of the effective potential term. One of the main reasons behind the specific approximations made for the kinetic and effective potential terms is, of course, that the resulting Hamiltonian (32) is separable and analytically solvable. $\mathbf{R}^2$ will be diagonalized simultaneously with H_{eff}, hence we can replace $\mathbf{R}^2$ by its eigenvalue R^2_χ, where χ refers to all quantum numbers which appear in this eigenvalue. H_{eff} then reduces to the Hamiltonian of a harmonic oscillator in ϱ with continuous angular momentum, and the spectrum is

$$E^\chi_{n_\varrho} = \sqrt{\frac{C_\varrho}{B_\varrho}}\Big(2n_\varrho + \nu + 4\Big) \tag{33}$$

with $n_\varrho = 0, 1, 2, \ldots$ and

$$\nu = \sqrt{R^2_\chi + \frac{1}{4}} - 3\,. \tag{34}$$

Plugging in the eigenvalues for the different operators appearing in $\mathbf{R}^2$ in (31), we obtain the more explicit expression

$$E^{(\omega_1\omega_2\omega_3),\mathcal{L}}_{n_\varrho,N_2} = \sqrt{\frac{C_\varrho}{B_\varrho}}\Bigg\{2n_\varrho + \Bigg[4\sqrt{\frac{C_2}{B_2}}\sigma^2 N_2 + k_1(\omega_1+\omega_2+\omega_3)(\omega_1+\omega_2+\omega_3+12) + k_2(\lambda^2+\lambda\mu+\mu^2+3\lambda+3\mu) + k_3\mathcal{L}(\mathcal{L}+1) + k_4\Bigg]^{\frac{1}{2}} - \sqrt{k_4}\Bigg\}\,, \tag{35}$$

where N_2 is the quantum number of the five-dimensional harmonic oscillator, hence $N_2 = 0, 1, 2, \ldots$. As we are interested only in energy differences, we have subtracted in (35) the eigenvalue for the vacuum, which has the quantum numbers $n_\varrho = 0$, $N_2 = 0$ and $\omega_k = 0$ $(k = 1, 2, 3)$, consequently $\lambda = \mu = \mathcal{L} = 0$ from (2), (3) and (30). We see explicitly from (35) that k_4 is constrained to take non-negative values.

In our ansatz we have introduced a total of eight independent parameters which have to be adjusted in order that H_{eff} in (32) approximate as closely as possible the exact effective Hamiltonian. Lacking knowledge of the latter, we will use information from lattice simulations for the same purpose. In the next section, we will determine the values of the six parameter combinations

$$\sqrt{\frac{C_\varrho}{B_\varrho}},\ \sqrt{\frac{C_2}{B_2}},\ k_1,\ k_2,\ k_3,\ k_4\,, \tag{36}$$

appearing in (35), by fitting the spectrum of H_{eff} to the glueball spectrum as measured in quenched lattice QCD.

Finally, we will give explicit expressions for the eigenstates of H_{eff}. Following [12], we have

$$|\, n_\varrho N_2, \Lambda t L, JM, \Omega\mathcal{L}, (\omega_1\omega_2\omega_3)\delta\,\rangle = \mathcal{F}^\nu_{n_\varrho}(\varrho)F^\Lambda_{(N_2-\Lambda)/2}(\beta)\sum_{K'}\Phi^{\Lambda t L}_{K'}(c) \times\sum_{K,\mathcal{K}}(JK,\mathcal{L}\mathcal{K}\mid LK')D^J_{KM}(\theta)D^{(\omega_1\omega_2\omega_3)}_{\Omega\mathcal{L}\mathcal{K};\delta(0,0)}(\phi)\,. \tag{37}$$

$\mathcal{F}^{\nu}_{n_\varrho}$ are the eigenfunctions of the one-dimensional harmonic oscillator in ϱ given explicitly by

$$\mathcal{F}^{\nu}_{n_\varrho}(\varrho) = \left[\frac{2(B_\varrho C_\varrho)^{2+\nu/2}\, n_\varrho!}{\Gamma(n_\varrho+\nu+4)}\right]^{\frac{1}{2}} \varrho^{\nu+7/2} L^{(\nu+3)}_{n_\varrho} \times \left(\sqrt{B_\varrho C_\varrho}\,\varrho^2\right) e^{-\sqrt{B_\varrho C_\varrho}\,\varrho^2/2}\,, \tag{38}$$

where $L^{(\alpha)}_n$ is a Laguerre polynomial and ν was defined in (34). The function F in (37) is the radial part of the five-dimensional harmonic oscillator and Φ is the part that depends on c. Both are given in [23]. D^J_{KM} represents the Wigner D-function [20] whilst the other D-function is the representation matrix of $O(8)$ in the approximation of large σ^2. Λ has the meaning of seniority for the five-dimensional harmonic oscillator and gives the number of $J = 2$ gluon pairs not coupled to spin zero. The angular momentum L is restricted to the values allowed for a five-dimensional harmonic oscillator [23], which in particular has no $L = 1$ state. The parameter t is a multiplicity index appearing in the classification of the five-dimensional oscillator and Ω plays the rôle of the K in (2) and (3) for the reduction of $SU(3)'$ to $SO(3)'$. Finally, δ is a further multiplicity index in the reduction of $O(8)$ to $SU(3)$ colour.

The basis given in (37) is one of the possible realizations of the general expression given in (5). In principle the direct relation could be found by using Dragt's theorem [28] which relates the representation of states in terms of boson creation operators to the coordinate representation. We can in particular identify excitations of the one-dimensional oscillator (in ϱ) and the five-dimensional oscillator (in β, c) with gluon pairs created by the operators in (4). The relation is given through (17)–(21) and involves the parameters B_ϱ, C_ϱ and B_2, C_2, respectively. The total angular momentum of the pairs with quadrupole momentum is given by L (the other pairs do not contribute), where the allowed values of L correspond to the fact that these gluon pairs represent identical bosons. The minimum weight states in (5) transforming according to a certain $O(8)$ irrep $(\omega_1\omega_2\omega_3 0)$, are described in (37) by the corresponding $O(8)$ representation matrix. We can interprete the value of $\mathcal{L}$ as the angular momentum of these states, as is evident from considering a state with $L = 0$, where $J = \mathcal{L}$ by the Clebsch-Gordan coefficient in (37). In general, the total angular momentum J of a state arises as the (angular momentum) sum of L stemming from the $J = 2$ pairs and $\mathcal{L}$ originating from the minimum weight state. Interpreting the dependence of the energy in (35) on the quantum numbers of the states, the tensor gluon pairs and the unpaired gluons (described by the minimum weight states) interact strongly, while the scalar gluon pairs appear in our approximation as free particles.

Using the basis (37), we can calculate the expectation value of ϱ^2 which turns out to be

$$\langle\varrho^2\rangle = \frac{1}{C_\varrho} E^{\chi}_{n_\varrho}\,, \tag{39}$$

with $E^{\chi}_{n_\varrho}$ taken from (33), i.e. including the vacuum energy. Equation (39) shows that the energy of a state is determined exclusively by the square of the intensity of the gluon field. Let us emphasize again that our model permits us only to calculate the spatial field distribution for zero-momentum states of glueballs, therefore we have no information about the internal structure of the states.

We are confident that the Hamiltonian presented in (32) can serve as a first approximation to the exact effective Hamiltonian. In future developments the basis (37) will then play the important rôle of a perturbative basis in which to diagonalize the improved effective Hamiltonian. However, as there is at present no quantitative way to link our model to fundamental pure QCD it cannot be regarded as more than an ansatz somewhat in the spirit of a Landau-Ginzburg expansion of the free energy in statistical physics. We would claim that it is more concretely rooted in the fundamental microscopic theory than the former as our order parameter is directly related to the fundamental field content of QCD whereas, for example in superconductivity, the Landau-Ginzburg ansatz does not relate the scalar order parameter to the underlying electron degrees of freedom.

4 The glueball spectrum

The values of the parameters in (35) will be determined by adjusting the spectrum of our model to the masses of the 16 glueball states given by the lattice calculation of [9]. For the three lowest-lying states we use the values cited in [29] (rather than [9]) obtained as weighted averages over several lattice measurements done by different groups (see the references in [29]). The 0^{--} glueball state comes with a huge error in [9], so we decided to take the value from [30] instead, where a lattice Hamiltonian method was used. Finally, we use some of the (preliminary) values of [10] for the fit, which result from a new generation of glueball measurements on anisotropic lattices (see also [31]). We have employed the CERN-MINUIT fitting routine [32] in order to obtain the best overall agreement. The resulting fit is given by the following formula (cf. (35))

$$\begin{aligned} E^{(\omega_1\omega_2\omega_3),\mathcal{L}}_{n_\varrho,N_2} &= 0.805\,\mathrm{GeV} \\ &\times\Big\{2n_\varrho + \Big[1.17\sigma^2 N_2 + 0.165(\omega_1+\omega_2+\omega_3) \\ &\times(\omega_1+\omega_2+\omega_3+12) + 0.877(\lambda^2+\lambda\mu+\mu^2+3\lambda+3\mu) \\ &\times + 0.028\mathcal{L}(\mathcal{L}+1)\Big]^{\frac{1}{2}}\Big\}. \end{aligned} \tag{40}$$

Observe that the best value for the parameter k_4 is zero, so effectively we have only used five parameters. Note furthermore that the dependence on J arises only indirectly through N_2, $\mathcal{L}$ and the restriction given by the $SU(2)$ Clebsch-Gordan coefficients in (37).

In Table 2 we list the results of the fit for the states considered in [9]. For states marked with a cross only upper limits have been given in [9]. Before commenting on the quantitative aspects of the fits, however, let us emphasize that *the overall ordering of glueball states is to a*

Table 2. Glueball masses calculated within the model compared to lattice calculations. The values of the parameters used in our model are given in (40). See Sects. 2 and 3 for our classification of the glueball states displayed in the first four columns. Columns five and six give the usual J^{PC} classification along with the value of the mass determined from our model Hamiltonian. The lattice results which have been used to fit the parameters are given in the last two columns. While the next to last column cites the best confirmed values currently available, the last column refers to preliminary results of a new generation of glueball calculations on anisotropic lattices (we did not include the value for the 0^{++} glueball due to problems with this specific state in the new method). The errors displayed refer exclusively to statistical errors, while there might be an additional overall variation in the scale. In particular, the scales used in the last two columns may slightly differ from one another. Values which are only upper limits, are marked by a cross. Some of the lattice values have actually been read off the figures in the respective publications, hence the data given might be erroneous in the last digit

$(\omega_1\omega_2\omega_30)$	$\mathcal{L}$ (λ,μ)	n_ϱ	N_2	J^{PC}	mass [GeV]	lattice [Ref.]	[10] (prelim.)
(0000)	0 (0,0)	1	0	0^{++}	1.61	1.61±.03 [29]	–
(0000)	0 (0,0)	0	1	2^{++}	2.13	2.23±.22 [29]	2.39±.01
(0000)	0 (0,0)	0	3	3^{++}	3.69	3.92±.48 [9]	3.69±.04
(2200)	2 (0,2)	0	1	1^{++}	4.50	3.96±.31 [9]	4.12±.05
(1110)	0 (0,0)	0	0	0^{-+}	2.19	2.23±.37 [29]	2.59±.03
(1110)	0 (0,0)	0	1	2^{-+}	3.41	3.01±.18 [9]	3.07±.02
(3220)	1 (1,0)	0	0	1^{-+}	4.07	× 3.71±.39 [9]	4.18±.03
(1110)	0 (0,0)	0	3	3^{-+}	5.03	× 5.83±.66 [9]	4.67±.05
(2110)	1 (1,0)	0	0	1^{+-}	3.03	2.90±.26 [9]	2.94±.02
(2110)	1 (1,0)	0	1	2^{+-}	4.09	3.89±.66 [9]	4.10±.04
(2110)	1 (1,0)	0	1	3^{+-}	4.09	× 6.18±.89 [9]	3.54±.02
(4220)	0 (2,0)	0	0	0^{+-}	4.77	× 2.99±.75 [9]	4.74±.05
(2210)	1 (0,1)	0	0	1^{--}	3.38	4.36±.48 [9]	3.85±.04
(3110)	0 (2,0)	0	0	0^{--}	3.84	3.93±.48 [30]	4.94±.05
(3110)	2 (2,0)	0	0	2^{--}	3.86	3.94±.35 [9]	3.93±.02
(3000)	3 (3,0)	0	0	3^{--}	3.91	× 5.74±.89 [9]	4.13±.08

large extent independent of the choice of the model Hamiltonian and relies only on the assumption that the number of constituent gluons is approximately a good quantum number and that more gluons correspond to higher energy. Of course, it is not clear a priori that the concept of a constituent gluon is a sensible one. However, in our approximation the quantity $N = h_1 + h_2 + h_3 = 2(n_\varrho + N_2) + \omega_1 + \omega_2 + \omega_3$ is a good quantum number and can be interpreted as the number of constituent gluons. It should be appreciated that the level ordering of the glueball spectrum does not follow any easily recognizable pattern, as was already stressed in [9]. Using the above "minimal" assumptions, however, we already obtain good agreement with the ordering observed on the lattice. The two states that come out relatively too low, 1^{--} and 3^{--}, will be shifted upwards by large corrections stemming from the Casimir $\mathcal{C}_2(\lambda,\mu)$ of the $SU(3)'$ subgroup of the $U(3)'$ group related to $O(8)$ (discussed in the previous section). Further below we will compare our results with those of other effective models, such as the MIT bag or the flux-tube model.

Let us now proceed to a more quantitative comparison of our results with the lattice measurements. It must be noted that the overall agreement is surprisingly good, considering the simple-mindedness of our model. The most significant deviation of our values from the results of [9] occurs for the 0^{+-} state. However, the statistical error of the lattice measurement is probably too large for this discrepancy to be taken seriously. Furthermore, our value is in excellent agreement with the more recent calculation of [10]. As far as the latter is concerned, the statistical errors given in [10] are remarkably small, so if they are taken seriously, deviations from our values are more clearcut in this case. There is, however, an uncertainty in the overall energy scale for the measurements (not included in the errors cited here), so there might be a proportionality factor involved when going from one set of lattice measurements to another. Comparing our results to those of [10] we find appreciable deviations for the 3^{+-} and 1^{--} states, but most importantly for the 0^{--} where the deviation is twice as large as in all other cases. Our value is however in excellent agreement with the central value given in [30] (after normalizing to the mass of the 0^{++} according to [29]).

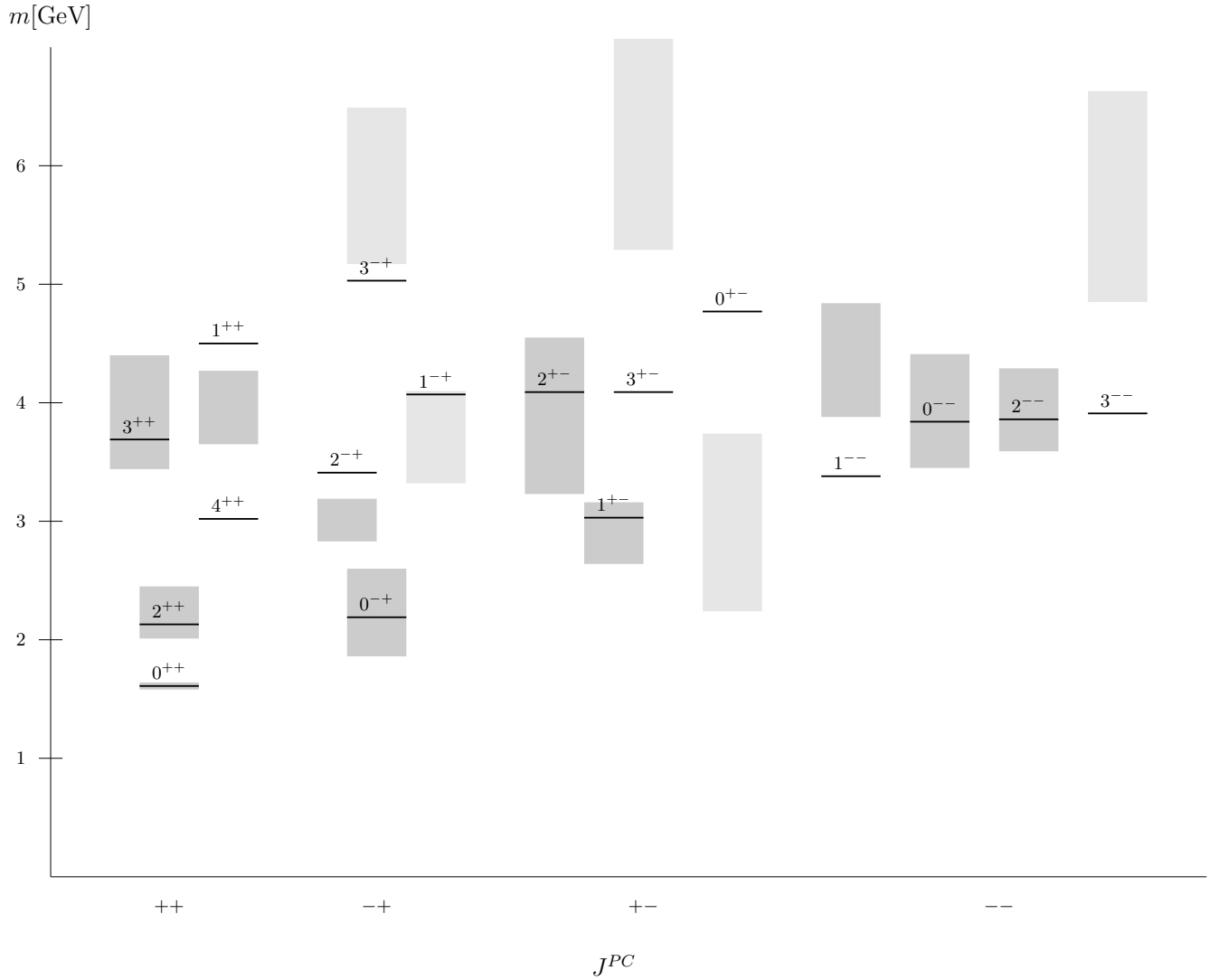

Fig. 1. Comparison of our predictions and current "best values" for the glueball masses as given in Table 2 (columns six and seven), arranged according to their PC eigenvalues. Our results are indicated by solid lines, while the lattice values, including statistical errors (one-sigma deviations), are represented by grey bars. Light grey bars indicate upper limits. More than two-sigma deviations occur for the mass of the 1^{++} versus our prediction of the 4^{++} (see the discussion in Sect. 4) and for the 0^{+-}, where our value is however in excellent agreement with the data of [10] (see Table 2)

Of course, deviations from our results are to be expected, since our model can only be considered as a first approximation to the exact effective Hamiltonian obtained from QCD by integrating out all non-constant modes. In particular, the form of the potential term (see the discussion preceding (32)) is merely guessed at. As a consequence, one expects the degeneracy of several states in our model to be lifted by the correponding corrections. The good qualitative agreement with lattice results, however, leads one to believe that those corrections are not so large as to render our approximation meaningless.

In Table 3 we have gathered the predictions from our model for states that may be accessible to future lattice measurements. Included are three excited states that have already been measured on the lattice [10], the results being in good agreement with our predictions. On a (hyper)cubic lattice one has to consider that rotational symmetry is broken down to invariance under the cubic group O. There exist only five irreps under spatial "rotations", termed A_1, A_2, E, T_1 and T_2. Consequently, the identification of a state on the lattice with a state in the continuum having angular momentum J is ambiguous and relies partially on the assumption that glueballs with higher J are heavier than those with smaller J, an assumption that is obviously not always fulfilled. Tables listing the O irreps contained in a given $SO(3)$ irrep are available for example in [33].

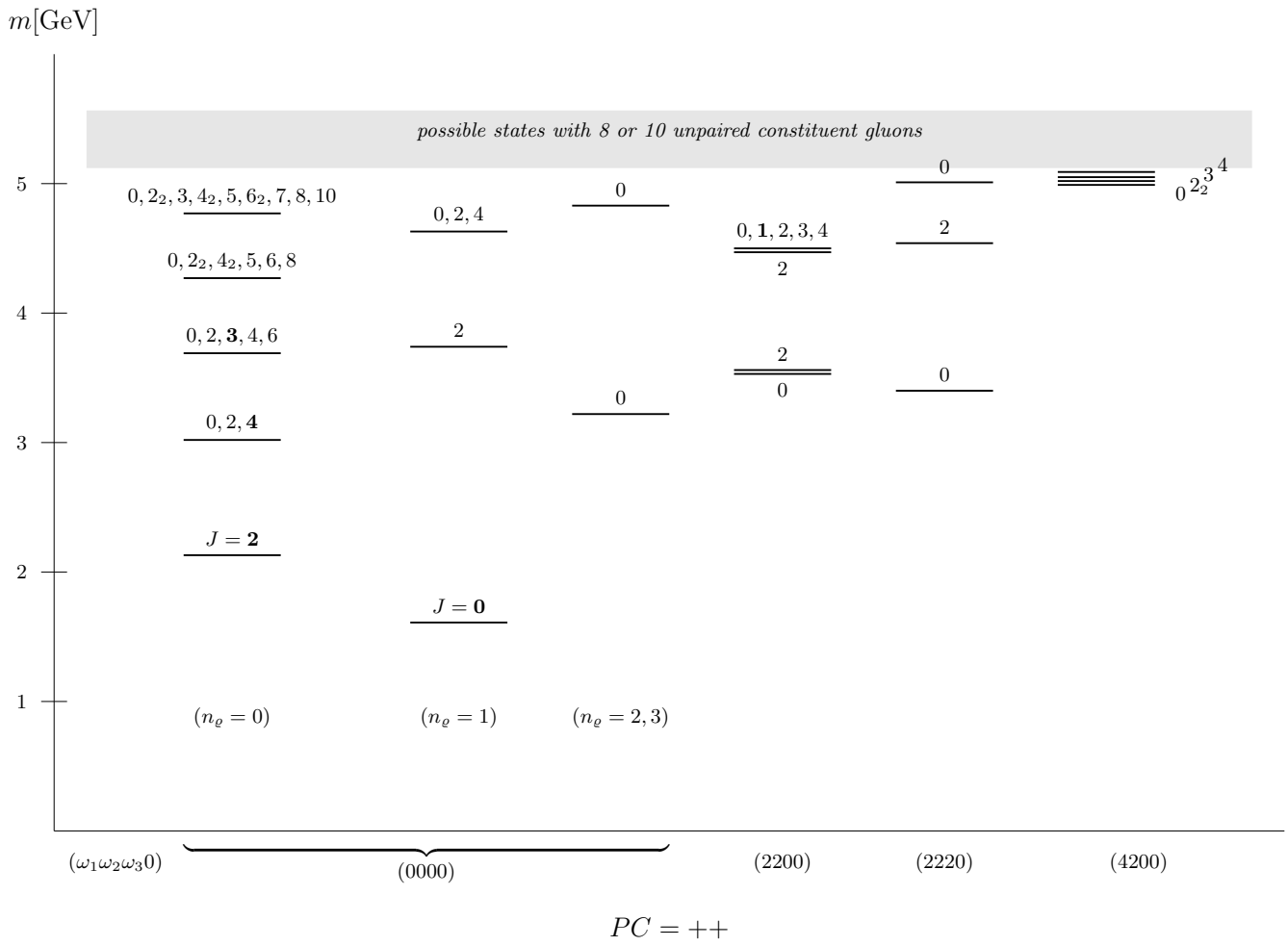

Fig. 2. Our prediction for the complete glueball spectrum with $PC = ++$ below 5.08 GeV, using the parameters in (40). Note how dense the spectrum is in comparison to what one would see on a lattice (Table 3). States listed in Table 2 and Fig. 1 are marked by bold face letters. The lowest-lying state is, of course, the vacuum (not indicated in the figure) which has $(\omega_1\omega_2\omega_3 0) = (0000)$, $(\lambda, \mu) = (0, 0)$, $\mathcal{L} = 0$ and $n_\varrho = N_2 = 0$ and hence $J^{PC} = 0^{++}$ and whose energy has been set to zero. Above 5.08 Gev, there are possible states with at least 8 unpaired constituent gluons ($\omega_1 + \omega_2 + \omega_3 \geq 8$). The charge conjugation eigenvalues C of these states are as yet unknown

State-of-the-art lattice calculations (on anisotropic lattices with improved actions) are able to determine the energies of the ground states and some first excited states for any given O irrep [10]. Table 3 gives a list of our predictions for the energies of the first and second excited states for all R^{PC} (R an irrep of O). Due to the degeneracy of several states in our model, to be lifted by the inclusion of corrections to our ansatz for the effective Hamiltonian, the degree of excitation of a certain state (especially on the lattice) cannot always be determined from our mass estimates. In fact, these corrections together with mass shifts due to possible mixing of different states with the same J^{PC} may change the sequence of states substantially in energy regions with a high density of states (see Figs. 2–4). Our predictions for the level ordering in these regions should be taken with a grain of salt.

We would like to point out that within our model we obtain a 4^{++} state lying considerably lower in mass than the 1^{++}. Given that both $J = 1$ and $J = 4$ contain an irrep T_1 of the cubic group, on the lattice the 4^{++} would have been mistaken, had it indeed been found, for a 1^{++} according to the above-mentioned rule of thumb. Hence our value differs by more than two standard deviations from the result of [9], the latter being consistent with the more recent calculation in [10].

The states that are marked by an asterisk in Table 3 are only accessible on the lattice by measuring second excited states in the corresponding R^{PC} sector, a technical challenge which has as yet not been met. More easily accessible states, to be compared with our predictions, are $T_1^{-+*}, A_2^{+-*}, E^{+-*}, A_2^{--*}, E^{--*}$ and T_1^{--*} corresponding to the continuum states $4^{-+}, 3^{+-*}, 2^{+-*}, 3^{--*}, 2^{--*}$ and 1^{--*}, respectively, with the predicted massses given in

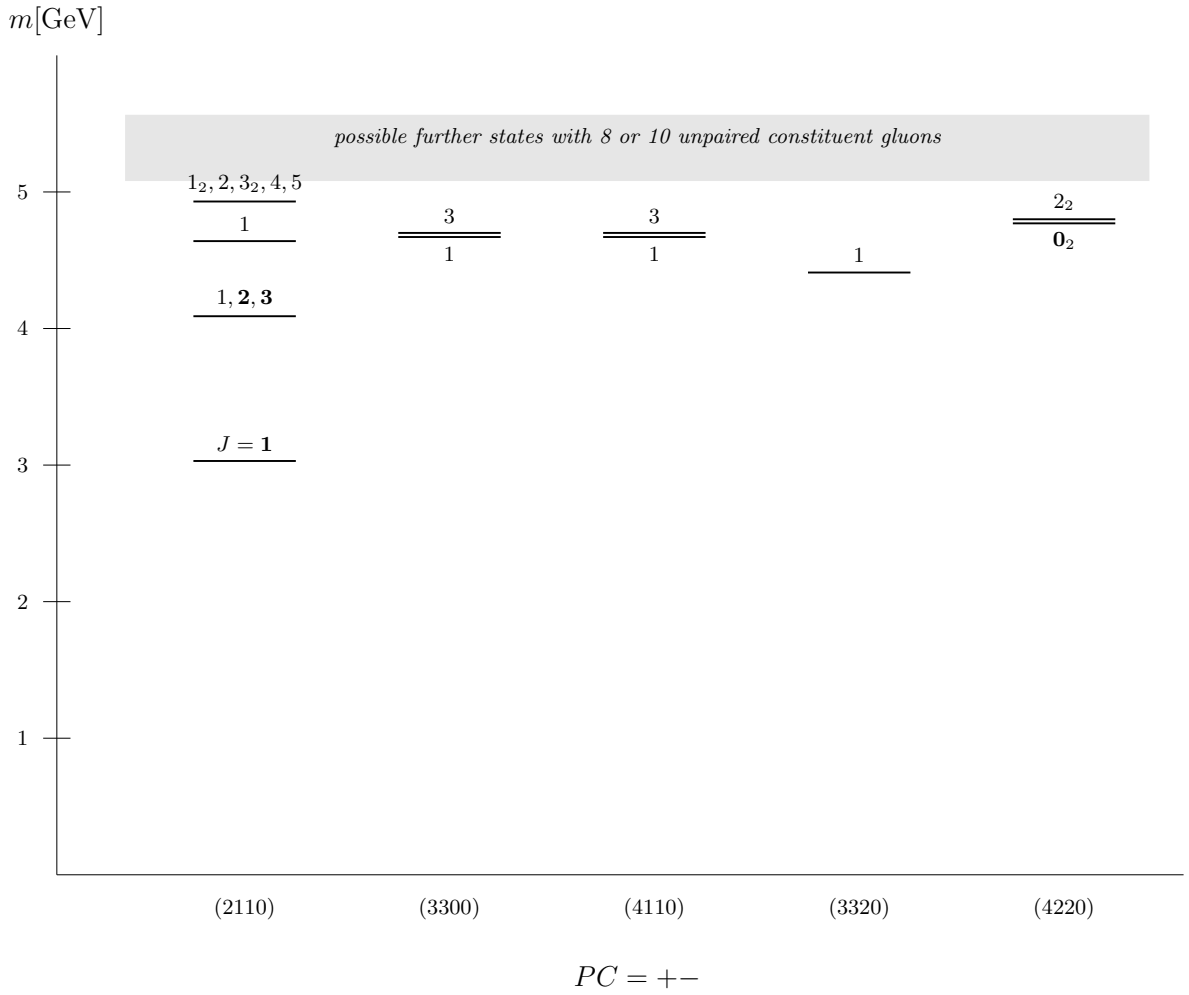

Fig. 3. As in Fig. 2, but for $PC = +-$. Note that the states with $(\omega_1\omega_2\omega_3 0) = (3320)$ and (4220) have 8 unpaired constituent gluons. In these particular cases we have determined the charge conjugation to be $C = -1$

Table 3. We should also mention that there are particular problems with the mass determination of 0^{++} states on anisotropic lattices, which is why we did not cite the corresponding results in Tables 2 and 3. Ignoring these difficulties, the values given by the collaboration of [10, 31] are $(1.63 \pm .03)$ GeV for the 0^{++} and $(2.84 \pm .04)$ GeV for the 0^{++*}. While the first value is consistent with the one given in Table 2 from [29], the second one is in good agreement with our prediction.

In Fig. 2 we graphically compare our fit with the values for the glueball masses as cited in [9,29,30]. The latter are represented by grey bars including the one-sigma deviation for the statistical errors. Light grey bars denote upper limits to the masses only. The other figures show our predictions for the full glueball spectrum up to a certain energy, obtained by using the parameters determined before. The states are ordered with respect to their PC eigenvalues and the $O(8)$ irreps.

The general prediction then is that the spectrum is *much denser* than seen on the lattice, in particular in the sector $PC = ++$. Notice, however, that there is no contradiction with the lattice results due to the fact that on the lattice, with few exceptions, only the lowest-lying state in a given representation R^{PC} is measured. A potential problem is the low-lying 4^{++} as discussed above.

Let us now discuss the present limits in energy for our prediction of the spectrum, represented by the light grey bars in Figs. 2–4. While the parity of any state is simply given by $P = (-1)^{\omega_1+\omega_2+\omega_3}$ according to (7), we are at present lacking a systematic determination of the charge conjugation eigenvalues C for states with more than six unpaired gluons, i.e. $\omega_1 + \omega_2 + \omega_3 > 6$ (there is work in progress on this matter [24]). We will consider first the sectors with positive parity $P = +1$ and thus states with an even number of constituent gluons. Above 5.08 Gev, there are possible states with at least eight unpaired constituent gluons, the charge conjugation eigenvalues C of which are as yet undetermined. We do know, however, that the charge conjugation of states with $O(8)$ irreps (3320) and (4220) is $C = -1$ (see Appendix B). The lowest-lying

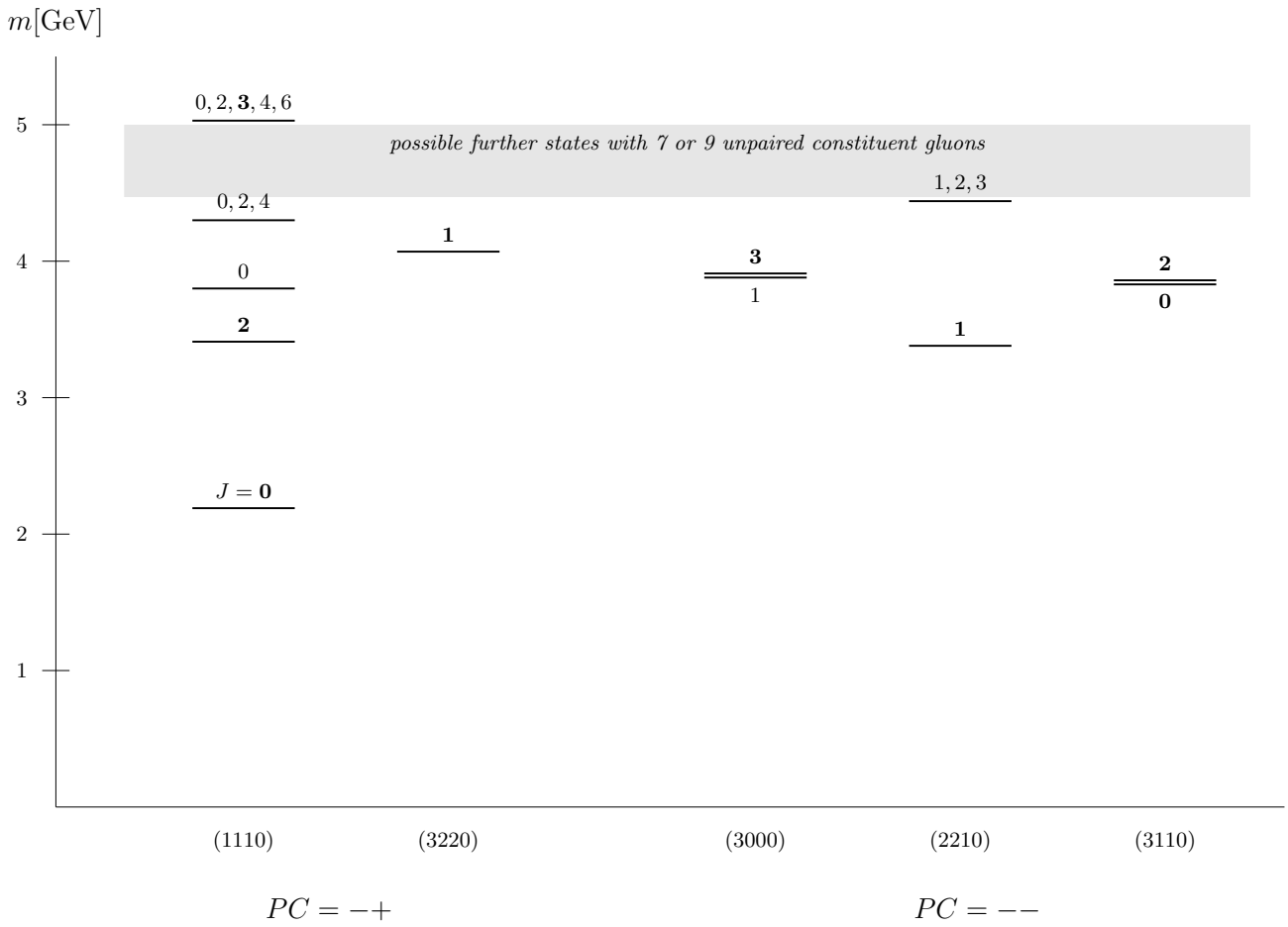

Fig. 4. As in Fig. 2, for states with $PC = -+$ and $--$ and masses below 4.46 GeV (with the exception of the $(\omega_1\omega_2\omega_3 0) = (1110)$ states at 5.03 GeV, see Sect. 4). Between these two values, there are states with at least 7 unpaired constituent gluons, whose charge conjugation eigenvalues have yet to be determined. We do know, however, the charge conjugation of states with $(\omega_1\omega_2\omega_3 0) = (3220)$ to be $C = +1$

of the C-undetermined states has $(\omega_1\omega_2\omega_3 0) = (4330)$, i.e. 10 unpaired constituent gluons, $(\lambda, \mu) = (1, 0)$, $\mathcal{L} = 1$ and $n_\varrho = N_2 = 0$, leading to $J^P = 1^+$ and a mass of 5.08 GeV. This state sets the limit indicated in Figs. 2 and 3.

Turning now to the negative-parity sectors, there are states with at least seven unpaired constituent gluons, whose charge conjugation eigenvalues have yet to be determined. The lowest-lying among them has $(\omega_1\omega_2\omega_3 0) = (3310)$, $(\lambda, \mu) = (0, 2)$, $\mathcal{L} = 0$ and $n_\varrho = N_2 = 0$, and hence $J^P = 0^-$ and a mass of 4.46 GeV. All states with nine unpaired constituent gluons have masses above that value. We have, however, determined the charge conjugation of (3220) to be $C = +1$ and that of (4210) (not indicated in the figure) to be $C = -1$. Using the latter fact one can show that the lowest-lying 3^{-+} state has a mass of 5.03 Gev, although there may be states with $PC = -+$ and other angular momenta between 4.46 and 5.03 GeV that we cannot classify completely at present. However, none of these states can contain a representation T_1 or A_2 on the lattice (i.e. $J \neq 1, 3, 4, \ldots$). Similarly, we can conclude that the first state in this energy range containing A_2^{--} is the 3^{--**} at 4.68 GeV. However, there are candidate states for A_1^{--*}, A_1^{--**} and E^{--**} below this value, so for the time being we cannot make predictions for these states on the lattice. All these facts have been used in constructing Tables 2 and 3.

Finally, we would like to compare our results to those of other effective models [1]–[5]. The oldest and also the most successful among these is the MIT bag model [1]. All predictions for glueball masses in this model that we were able to find in the literature come out far too low. If we, however, take the predictions of Donoghue, Johnson, Li (and of Jaffe, Johnson for the $(TM)^2$ modes) [1] and simply scale the energy up by a factor of about 2.2, the lattice results are roughly reproduced. At this level, fine-structure corrections are absent, and the quality of the approximation is comparable to our model when we just order the states in accordance with the number of constituent gluons.

Multiplying the energies by a factor of 2.2 roughly corresponds to changing $B^{1/4}$ to $2.2\,B^{1/4}$, where B is the bag constant that characterizes the confining force. In the old

Table 3. Predictions from our model for states that can possibly be measured on the lattice in the near future. The first five columns are as in Table 2, while the sixth gives the corresponding representations on the lattice, where the angular momentum J is replaced by the label R referring to the representations of the cubic group O (see [33]). A superscript asterisk marks an excited state in columns five and six (in column six we have listed only those components which can be seen on the lattice as at most second excitations). Where the degree of excitation of a certain state could not be determined from our mass estimates due to the degeneracy of several states in our model, possible higher degrees of excitation were indicated by superscript asterisks in parenthesis. The last column gives our mass estimates and a few preliminary results from the lattice. An asterisk in front of a value indicates that the corresponding state can only be determined from a second excitation on the lattice, which appears to be rather demanding with present techniques

$(\omega_1\omega_2\omega_3 0)$	$\mathcal{L}$ (λ,μ)	n_ϱ	N_2	J^{PC}	R^{PC}	mass [GeV]
(0000)	0 (0,0)	0	2	0^{++*}	$A_1^{++*(*)}$	3.02
(0000)	0 (0,0)	0	2	2^{++*}	$E^{++*(*)}, T_2^{++*(*)}$	3.02 3.29±.02 [10]
(0000)	0 (0,0)	0	2	4^{++}	$A_1^{++*(*)}, E^{++*(*)}, T_1^{++}, T_2^{++*(*)}$	3.02
(0000)	0 (0,0)	0	3	4^{++*}	$T_1^{++*(**)}$	3.69
(0000)	0 (0,0)	0	3	6^{++}	$A_2^{++(*)}, T_1^{++*(**)}$	3.69
(0000)	0 (0,0)	0	4	6^{++*}	A_2^{++**}	$*$ 4.27
(1110)	0 (0,0)	1	0	0^{-+*}	A_1^{-+*}	3.80 3.64±.04 [10]
(1110)	0 (0,0)	0	2	0^{-+**}	$A_1^{-+**(*)}$	$*$ 4.30
(1110)	0 (0,0)	0	2	2^{-+*}	$E^{-+*(*)}, T_2^{-+*(*)}$	4.30 3.93±.02 [10]
(1110)	0 (0,0)	0	2	4^{-+}	$A_1^{-+**(*)}, E^{-+*(*)}, T_1^{-+*}, T_2^{-+*(*)}$	4.30
(1110)	0 (0,0)	0	3	4^{-+*}	$T_1^{-+**(**)}$	$*$ 5.03
(1110)	0 (0,0)	0	3	6^{-+}	$A_2^{-+(*)}, T_1^{-+**(**)}$	5.03
(2110)	1 (1,0)	0	1	1^{+-*}	$T_1^{+-*(*)}$	4.09
(3300)	3 (0,3)	0	0	$3^{+-*(*)}$	$A_2^{+-*(*)}, T_2^{+-**(*)}$	4.69
(4110)	3 (3,0)	0	0	$3^{+-*(*)}$	$A_2^{+-*(*)}, T_2^{+-**(*)}$	4.69
(4220)	0 (2,0)	0	0	0^{+-*}	A_1^{+-*}	4.77
(4220)	2 (2,0)	0	0	2^{+-*}	E^{+-*}	4.79
(4220)	2 (2,0)	0	0	2^{+-**}	E^{+-**}	$*$ 4.79
(2110)	1 (1,0)	0	2	4^{+-}	A_1^{+-**}	$*$ 4.93
(3000)	1 (3,0)	0	0	1^{--*}	T_1^{--*}	3.88
(2210)	1 (0,1)	0	1	2^{--*}	$E^{--*}, T_2^{--**(*)}$	4.44
(2210)	1 (0,1)	0	1	3^{--*}	$A_2^{--*}, T_2^{--**(*)}$	4.44
(3000)	1 (3,0)	0	1	3^{--**}	A_2^{--**}	$*$ 4.68

bag model calculations this constant has been adjusted to the meson spectrum. It then appears plausible that $B^{1/4}$ has to be changed by a factor of $2.2 \approx 9/4$ in order to describe glueballs, since the factor 9/4 can be understood as the ratio of the colour charge of gluons to that of quarks. Of course, the rather large fine-structure corrections would have to be calculated anew to see if the MIT bag model can really account for a quantitative description of the glueball spectrum. As a first step, however, the masses of higher states should be estimated in order to make sure that the part of the spectrum above ~ 4 GeV as measured on the lattice can also be qualitatively understood within this model. The calculations in [15] can be taken as a hint that one will have to consider states with more than three gluons.

Let us remark that the constituent gluons of the bag model are very different from the ones in our model as they basically arise by putting free gluons into the bag. Consequently, boundary conditions at the bag surface and possible orbital motions play an important rôle. None of this makes any sense in our effective Hamiltonian description. A potential advantage of the bag model is that it allows, at least in principle, for a description of the inner structure of the glueballs.

In [34] it was suggested to consider the canonical dimension of the gauge-invariant operators that can create a glueball with a given J^{PC} out of the vacuum in order to obtain a qualitative understanding of the level ordering of the spectrum. The results are somewhat similar to the bag model (without fine structure corrections), but compare less favourably with the lattice data. Of course, it is also unclear how these ideas could be refined to yield quantitative predictions for the spectrum.

There are other models like the "glue-lego" model [2], the flux-tube model [3] and the non-relativistic potential model [4] which are, at least in their original forms, more or less excluded by the available lattice data. A relativistic version of the potential model is given by a recent Bethe-Salpeter calculation with massive constituent gluons [5]. With a gluon mass of 0.4 GeV very reasonable values for the masses of the $0^{++}, 2^{++}$ and 0^{-+} glueballs are obtained, while the 2^{-+} comes out far too low and may be spurious. In [6] a phenomenological QCD Hamiltonian is combined with a BCS type vacuum, which leads to an effective mass for the constituent gluons. Results are given for the masses of the 0^{++}, 2^{++}, 0^{-+} and 2^{-+} glueballs including excited states. The agreement with lattice data is satisfactory considering the approximations made. We emphasize that again the constituent gluons in all these models are different in nature from the ones in our model, and there is at present no way to relate the two concepts.

Finally, sum rule calculations [35] have their place somewhere between effective models and microscopic QCD. Up to now, predictions have been obtained only for the three lowest-lying glueballs and these are consistent with lattice measurements.

In some of the phenomenological models a 1^{-+} state with a relatively low mass is predicted. This state is of some importance for experimental glueball searches, since it cannot be obtained from the quark model for mesons. In our model the 1^{-+} is a seven-gluon state and consequently its mass is rather large, in agreement with the lattice data. The problem with this state in other models can be traced to additional longitudinal degrees of freedom for the massive gluons, which in our case are absent as we have already pointed out in Sect. 3.

We would like to conclude this section by remarking that in our model states with $n_\varrho \geq 1$ (with the exception of the 0^{++} glueball) appear as unbound states of n_ϱ 0^{++} glueballs and the rest, insofar as the total energy of the state is just the sum of the energies of these components. Within the model, this is of course merely a consequence of the fact that the ϱ-dependence of the effective Hamiltonian is given by a harmonic oscillator. In physical terms, such states may correspond to very loosely bound states of n_ϱ 0^{++} glueballs and other glueballs. On the lattice, these states may possibly be identified by analyzing their overlaps with different lattice operators and finite-volume effects. Such an enterprise is currently being envisaged [10]. From our model, there is precisely one candidate for such a weakly bound state that has already been measured on the lattice, namely the 0^{-+*} state which should appear as a bound state of a 0^{++} glueball with a 0^{-+} glueball. However, it must be noted that the lattice calculations of [10] give a larger mass for the 0^{-+} glueball and a slightly smaller one for the 0^{-+*}, so that the weak binding might be merely an artefact of our approximation.

5 Conclusions

In this paper we have performed a group-theoretic analysis of many-gluon states, based on the assumption that for the low-lying part of the spectrum only one mode of the gluonic field (transforming under colour rotations and parity transformations like a constant mode) is important. The classification of the states involves a $U(8)$ colour group and a $U(3)$ spin group. The former is reduced via an $O(8)$ group to the colour group $SU(3)$. For each given parity and charge conjugation the irreps of this $O(8)$ group classify bands of states. We remarked that in the context of charge conjugation the decomposition into elementary permissible diagrams (epds) proved very useful. The systematic determination of epds for higher $O(8)$ irreps is presently under investigation [24].

One of the outcomes of the analysis is the possibility of defining a certain quantum number which we identified with the number of constituent gluons, which again we emphasize are different from perturbative gluons or constituent gluons arising in other phenomenological models. The ordering of the states with respect to this quantum number allows for a qualitative explanation of the level ordering of glueball states with respect to spin as observed in lattice calculations, assuming that the energy of the states increases monotonically with the number of constituent gluons. The ordering of the glueball states has been one of the most puzzling results of lattice simulations. It is worth noting that the MIT bag model might also be able to reproduce the glueball spectrum, provided that the value of the bag constant is chosen differently from the one used for quark states. To check this assertion new bag model calculations for glueballs are needed.

In the second part of the paper we motivated a specific form for an effective Hamiltonian of low-energy pure QCD which depended only on the spatially constant modes of the gluon field. We introduced collective coordinates related to the monopole and quadrupole degrees of freedom of the intensity of the gluon field and rewrote the kinetic part of the Hamiltonian in terms of these variables using the results of [12]. Expanding in terms of b and $(1/\sigma^2)$ where b parametrizes the absolute deformation of the intensity distribution and σ^2 is related to the quantum numbers of the $O(8)$ group, and making an additional simplifying assumption for the potential term, we were able to deduce to lowest order a Hamiltonian which is a function of a one- and a five-dimensional harmonic oscillator and of Casimir operators contained in the $U(8) \supset O(8) \supset SU(3)$ chain of groups. Taking into account higher contributions via a redefinition of the coefficients of these operators we arrived at a phenomenological, QCD-motivated model Hamiltonian. The eigenstates of this Hamiltonian could be classified by the quantum numbers arising in the group-

theoretical analysis, and in particular the number of constituent gluons turned out to be a good quantum number.

By fitting the parameters appearing in the Hamiltonian we adjusted the spectrum of the model Hamiltonian to the "experimental" glueball spectrum, "experimental" in this context meaning lattice simulations. The general level of agreement was seen to be very good and predictions for several further states could be made. In qualitative terms, we predicted a much *denser* spectrum than that seen on the lattice without, however, running into inconsistencies with present lattice measurements. We have thus, with little input, principally the kinematical, i.e. group theoretical, structure of QCD, been able to deduce a great deal about the structure of an effective QCD Hamiltonian.

We can naturally envisage several possible further investigations. Of course, an obvious challenge is the derivation of an effective Hamiltonian from first principles. Without some form of infrared cutoff, such as a fermi-sized box, this will require a solution of the full crossover problem, i.e. a description of the interpolation between quarks and gluons as effective degrees of freedom at high energies and hadrons at low energies. The collective coordinates introduced in this paper may prove useful in this context for the description of the low-energy degrees of freedom. Furthermore, a theoretic justification of the basic assumption made, that an effective Hamiltonian depending only on the constant modes of the field can yield the full glueball spectrum, is highly desirable. One could also think of possible extensions of the model, for example by including composite operators, in order to describe the inner structure of the physical states. Finally, it would of course be of interest to try and define collective variables related to quarks and anti-quarks, in a similar fashion to what has been done here for gluons. In that way the model could be extended to describe quark-model mesons and baryons, as well as the mixing between quarkic mesons and glueballs. Work on these topics is in progress.

P.O.H. thanks Yu. Smirnov for stimulating discussions concerning the classification of many-gluon states and the DAAD and DGAPA for financial support. A.W. would like to thank C. Morningstar for communicating his latest results for the glueball spectrum from anisotropic lattice calculations. C.R.S. thanks P. van Baal for useful and interesting discussions. This research was partially funded by CONACyT grant number 3298P-E9607.

Appendix A Reduction of $O(8)$ to $SU(3)$

In this section will determine the reduction of the group $O(8)$ to $SU(3)$. The reduction from $U(8)$ to $SU(3)$ is known [15,19] as is the reduction from $U(8)$ to $O(8)$ [17]. The reduction from $O(8)$ to $SU(3)$ can then be obtained in a recursive manner, starting from the simplest irrep [0] in $U(8)$.

For the scalar irrep [0] of $U(8)$, the reduction to $O(8)$ is (0000) and the reduction to $SU(3)$ is $(0,0)$, hence the $SU(3)$-irrep $(0,0)$ is contained once in the (0000) irrep of $O(8)$, which is trivial. Also the corresponding reduction of $U(3)$ via $SU_J(3)$ to $SO(3)$ is trivial, i.e. the [0] irrep of $U(3)$ only contains the spin $J=0$. We have used the fact that the Young tableaux of the $U(8)$ and $U(3)$ irreps have to be the same.

The $U(8)$ irrep of one gluon (not presented in Table 1 because we give only those irreps of $U(8)$ which contain at least one scalar irrep of $SU(3)$) reduces to (1000) of $O(8)$ [17] and to $(1,1)$ of $SU(3)$, consequently the (1000) irrep of $O(8)$ must contain the $(1,1)$ irrep of $SU(3)$. For the corresponding reduction of $U(3)$ to $SO(3)$ we obtain $J=1$, the expected result for a one-gluon state.

For the case of a two-gluon system we have the symmetric [2] and antisymmetric $[1^2]$ irreps of $U(8)$. The first reduces to $SU(3)$ as $(2,2)+(1,1)+(0,0)$ (see [15,19]), and for the corresponding reduction of $U(3)$ to $SO(3)$ we find $J=0,2$, using the rules of (3). The reduction to $O(8)$ for [2] of $U(8)$ is given by (2000) + (0000) ([17]). The irrep (0000) already appeared in the reduction of the [0] irrep of $U(8)$, and we know that it contains one colour zero irrep of $SU(3)$. The other $SU(3)$ irreps appearing in the list of the reduction of $U(8)$ to $SU(3)$ are therefore contained in the (2000) irrep of $O(8)$, and in particular the latter cannot contain any colour scalar. For the antisymmetric irrep $[1^2]$ of $U(8)$ the reduction to $SU(3)$ yields $(1,1)+(3,0)+(0,3)$ according to [15,19], and the reduction to $O(8)$ is (1100), hence (1100) contains precisely the $SU(3)$ irreps given above.

In this manner, we can proceed recursively towards higher $U(8)$ irreps, thereby obtaining the complete reduction of $O(8)$ to $SU(3)$, and in particular the number of colour singlets contained in a given $O(8)$ irrep. The results for up to six constituent gluons are listed in Table 1, together with the possible values of J from the reduction to $SO(3)$.

Appendix B Elementary permissible diagrams

As claimed in Sect. 2, any state characterized by the quantum numbers in (1) can be obtained by a coupling of epds. Here we list all the epds with up to six gluons:

$$(2,[2],J_1) = [b_a^\dagger \times b_a^\dagger]^{[J_1]}_{J_1}$$

$$(3,[3],J_2) = d_{abc}\left[[b_a^\dagger \times b_b^\dagger]^{[2]} \times b_c^\dagger\right]^{[J_2]}_{J_2}$$

$$(3,[1^3],J_3) = f_{abc}\left[[b_a^\dagger \times b_b^\dagger]^{[1]} \times b_c^\dagger\right]^{[J_3]}_{J_3}$$

$$(4,[2^2],J_4) = \hat{Y}_{[2^2]} * d_{abc}d_{ab'c'} \times \left[[b_b^\dagger \times b_c^\dagger]^{[2]} \times [b_{b'}^\dagger \times b_{c'}^\dagger]^{[2]}\right]^{[J_4]}_{J_4}$$

$$(4,[21^2],J_5) = \hat{Y}_{[2,1^2]} * d_{abc}f_{ab'c'}[b_b^\dagger \times b_c^\dagger]^{[0]}[b_{b'}^\dagger \times b_{c'}^\dagger]^{[J_5]}_{J_5}$$

$$(5,[31^2],J_6) = \hat{Y}_{[3,1^2]} * f_{aa'a''}f_{abc}d_{a'b'c'} \times \left[\left[[b_b^\dagger \times b_c^\dagger]^{[1]} \times [b_{b'}^\dagger \times b_{c'}^\dagger]\right]^{[J_c]} \times b_{a''}^\dagger\right]^{[J_6]}_{J_6}$$

$$(5,[2^21],J_7) = \hat{Y}_{[2^2,1]} * f_{aa'a''}f_{abc}d_{a'b'c'}$$

$$\times\left[\left[[b_b^\dagger\times b_c^\dagger]^{[1]}\times[b_{b'}^\dagger\times b_{c'}^\dagger]\right]^{[J_c]}\times b_{a''}^\dagger\right]_{J_7}^{[J_7]}$$

$$(6,[3^2],J_8)=\hat{Y}_{[3^2]}*d_{a_1a_2a_3}f_{a_1b_1c_1}f_{a_2b_2c_2}f_{a_3b_3c_3}$$
$$\times\Big[[b_{b_1}^\dagger\times b_{c_1}^\dagger]^{[1]}\times[b_{b_2}^\dagger\times b_{c_2}^\dagger]^{[J_b]}$$
$$\times[b_{b_3}^\dagger\times b_{c_3}^\dagger]^{[J_c]}\Big]_{J_8}^{[J_8]}. \quad (41)$$

On the right-hand side the Young operator $\hat{Y}$ appears, which projects to a definite symmetry with respect to $U(3)$ (or $U(8)$) and hence via $[h_1h_2h_3]=[\omega_1\omega_2\omega_3]$ to a definite $O(8)$ irrep. The spins denoted by J_k $(k=1,2,...,8)$ are limited to the values $J_1=0,2$, $J_2=1,3$, $J_3=0$, $J_4=0,2$, $J_5=1$, $J_6=0,2$, $J_7=1$ and $J_8=1,3$. For the last three epds in (41) intermediate couplings with spin labels J_b and J_c appear, for each of which one of the possible values has to be chosen. These are examples of the ambiguity in the choice of epds mentioned in Sect. 2.

The epds have been determined by a procedure described in Appendix D using the reduction of $U(8)$ via $O(8)$ to $SU(3)$ as deduced by the methods of Appendix A. For the explicit construction of the epds we have used the fact that every colour singlet state can be built up from the three "fundamental tensors" δ_{ab}, f_{abc} and d_{abc} [36]. As a result, the decomposition of all states with up to six consituent gluons in epds is known, and the charge conjugations of the states can be obtained easily from the charge conjugations of the corresponding epds. The latter are readily determined from the explicit expressions (41) as shown in Appendix C. The results for the charge conjugations of all states with up to six gluons are given in Table 1, together with their parity eigenvalues determined by (7).

In order to reproduce the lowest 1^{-+} and 0^{+-} glueball states we have to consider minimum weight states with seven and eight gluons, respectively. The 1^{-+} state is contained in the $U(8)$ irrep $[32^2]$, which reduces to the $O(8)$ irreps (3220), (3000), (2200), and others that do not contain colour singlets. Among the former, (3000) and (2200) can be obtained via a product of already determined epds coupled to a definite $U(8)$ irrep. The (3220) irrep is given by a new epd,

$$(7,[32^2],J=1)=\hat{Y}_{[3,2^2]}*f_{aa_1a_2}d_{aa_3a_4}d_{a_1b_1c_1}d_{a_2b_2c_2}d_{a_3b_3c_3}$$
$$\times\left[\left[[b_{b_1}^\dagger\times b_{c_1}^\dagger]^{[J_1]}\times[b_{b_2}^\dagger\times b_{c_2}^\dagger]^{[J_2]}\right]^{[J_{12}]}\right.$$
$$\left.\times\left[[b_{b_3}^\dagger\times b_{c_3}^\dagger]^{[J_3]}\times b_{a_4}^\dagger\right]^{[J_{34}]}\right]_1^{[1]}, \quad (42)$$

which has parity $P=-1$ and charge conjugation $C=+1$ (see Appendix C).

The 0^{+-} state can be obtained by multiplying the two epds $(3,[1^3],0)$ and $(5,[31^2],0)$ coupled to the $U(8)$ irrep $[42^2]$ and the $O(8)$ irrep (4220). The charge conjugation of the state is the product of the charge conjugations of each epd, hence $P=+1$ and $C=-1$. In view of the later discussion of excited glueball states, we note that we have determined in similar ways the parity and charge conjugation of colour zero states in the $O(8)$ irrep (4210), contained in the $U(8)$ irrep [421], to be $PC=--$, and for the $O(8)$ irrep (3320) contained in $[3^22]$ we have obtained $PC=+-$.

Appendix C Charge conjugation

In order to determine the charge conjugation C of a state contained in a given $O(8)$ irrep $(\omega_1\omega_2\omega_30)$ (with $(0,0)$ as colour for its $SU(3)$ subgroup) we introduced in Sect. 2 the concept of epds. A list of epds for up to six gluons is given in (41). Having determined the decomposition of a state into epds, its charge conjugation can be obtained simply by multiplying the charge conjugations of the epds.

In order to deduce the properties of the epds under charge conjugation, we use the results in [21]. Consider the monomial

$$T_{a_1a_2.....a_n}b_{a_1}^{i_1\,\dagger}b_{a_2}^{i_2\,\dagger}...b_{a_n}^{i_n\,\dagger}, \quad (43)$$

where summation over the indices a_k is understood. Under charge conjugation the monomial transforms to

$$\eta_{a_1}\eta_{a_2}...\eta_{a_n}T_{a_1a_2.....a_n}b_{a_1}^{i_1\,\dagger}b_{a_2}^{i_2\,\dagger}...b_{a_n}^{i_n\,\dagger}, \quad (44)$$

the values of the η_a being given in (9).

As an example let us consider a special monomial of order four with colour zero and a certain angular momentum coupling,

$$d_{abc}f_{ade}\left[[b_b^\dagger\times b_c^\dagger]^{J_1}\times[b_d^\dagger\times b_e^\dagger]^{J_2}\right]_M^J. \quad (45)$$

Applying the charge conjugation operator and inserting a trivial factor $\eta_a^2=1$, the $SU(3)$ coupling $d_{abc}f_{ade}$ transforms to

$$(\eta_a\eta_b\eta_cd_{abc})(\eta_a\eta_d\eta_ef_{ade})\equiv d'_{abc}f'_{ade}. \quad (46)$$

Using (9) we note that

$$d'_{abc}=-d_{abc}$$
$$f'_{ade}=f_{ade}, \quad (47)$$

i.e. the d-symbol contributes a minus sign while the f-symbol contributes a positive sign.

This can be readily extended to any monomial with colour zero. All one has to do is count the number of d-symbols involved. Denoting this number by n_d, the charge conjugation is given by

$$C=(-1)^{n_d}. \quad (48)$$

The charge conjugations of the epds in (41) follow immediately.

Appendix D Construction of states

In this appendix, we will describe how states with up to six constituent gluons can be constructed from the epds listed

in (41). At the same time, we will show how to actually determine the epds with the help of the reduction of $U(8)$ via $O(8)$ to $SU(3)$ as given in Table 1.

We begin with the decomposition of a given state in a minimum weight state and gluon pairs. As stated in Sect. 2, the lowest $U(8)$ irrep containing a given $O(8)$ irrep $(\omega_1\omega_2\omega_3 0)$ is described by the Young tableau $[\omega_1\omega_2\omega_3]$. According to (5) we can build up all higher $U(8)$ irreps with the same $O(8)$ irrep by applying an arbitary coupling of $(n_1+n_2+n_3)$ pair operators to the lowest state. Each pair is represented by the $U(8)$ irrep [2], and is identical with the simplest epd $(2,[2],J_1)$ in the case $M_1 = J_1$ (maximum weight with respect to $SO(3)$).

A number of these pairs can be coupled to states represented by Young diagrams with an even number of boxes in each row. This restriction arises because the pairs themselves represent identical bosons with six degrees of freedom each. The problem of coupling of the pairs is equivalent to the reduction [17]

$$\begin{aligned} U(6) &\supset U(3) \\ [1] &\to [2] \\ [n] &\to \sum [2n_1, 2n_2, 2n_3]\,, \end{aligned} \tag{49}$$

where in the general reduction in the last line of (49) the sum is over different partitions of n, $n = n_1+n_2+n_3$. For example, we have for the coupling of two and three pairs

$$\begin{aligned} [2]\times[2] &= [4]+[2^2] \\ [2]\times[2]\times[2] &= [6]+[42]+[2^3]\,, \end{aligned} \tag{50}$$

respectively. The resulting irreps then have to be coupled with a minimum weight state, $[h_1h_2h_3] = [\omega_1\omega_2\omega_3]$.

We will now construct all the states appearing in Table 1 in this manner. Let us start by considering the (0000) irrep of $O(8)$. Coupling with one gluon pair operator, we obviously obtain the $U(8)$ irrep [2]. The results of the coupling of two and three pairs are given in (50). We thus get all the states with up to six gluons for the scalar irrep (0000) of $O(8)$ (cf. Table 1).

In Table 1 we find two $U(8)$ irreps with three gluons, [3] and $[1^3]$, both of which cannot be constructed by a coupling of pair operators. They hence give rise to epds, $(3,[3],J_2)$ and $(3,[1^3],J_3)$, which – in contradistiction to $(2,[2],J_1)$ – are minimum weight states. We now couple pair operators with these epds. As we exclusively consider states with up to six gluons, only one gluon pair will be coupled. We thus arrive at the $U(8)$ irreps $[5]+[41]+[32]$ for the $O(8)$ irrep (3000) and $[31^2]$ for (1110) in agreement with Table 1.

Similarly, there appear states with four and five gluons in Table 1, namely the $U(8)$ irreps $[2^2]$, $[21^2]$, $[2^21]$ and $[31^2]$, which obviously cannot be obtained through coupling of the epds with two or three gluons, and hence give rise to new epds themselves (see (41)). We can still couple one gluon pair with the four-gluon epds to obtain the states given in Table 1.

Finally, for the six-gluon states we consider couplings of the epds with three gluons as for example $[3]\times[3] = [6]+[42]$, where similar restrictions apply as for the coupling of several [2] irreps of $U(8)$ (see (106) of [17]). We hence obtain new minimum weight states for the $O(8)$ irreps (6000) and (4200). The other couplings are $[3]\times[1^3] = [41^2]$ and $[1^3]\times[1^3] = [2^3]$, with the corresponding minimum weight states (4110) and (2220).

There is one state in Table 1 which cannot be constructed in this way, namely the $U(8)$ irrep $[3^2]$, which consequently yields the last epd in (41). The decomposition of the states in epds is useful because it immediately determines the charge conjugation of the states in terms of the charge conjugation of the epds (see the previous Appendix C).

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